Symmetries in Bifurcation Theory: the Appropriate Context

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To my mathematical parents:
Isabel and Zé

Contents

Acknowledgements

The first version of my acknowledgements was getting too long and complicated. I was going backwards in time and just could not stop. It seems better to fix a starting point and go forward as much as I can. How about starting from when I was born?

My parents, Franklim and Alberta, became examples that I should always keep in mind.

– Hi mamã! Hi papá! Thanks for being exactly as you are. Thanks for being there whenever I need you.

Then a little brother came three years after me.

– We had a really good time together, didn't we Filipe?

I should also like to thank my big family and friends, especially the memory of my grandparents. They have always been active in the background of everything I do.

A very important lemma for me is that I always like what I am learning if I like the person that is teaching me. Clara Pacheco was one of my school teachers.

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At some point I went to do applied maths in Porto and I became interested in Bifurcation Theory. Isabel Labouriau was lecturing a course on it.

– Thanks Isabel! I did not hesitate much when you and Zé suggested that I come to Warwick and look for some more bifurcations, did I?

The first person that I met in England was Alberto. He helped me to settle here.

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Sometimes I felt a bit home sick and ran away to Portugal. I should like to thank Grupo de Matemática Aplicada do Porto, especially Isabel Labouriau and José Basto Gonçalves, for having me around at these moments. I had many encouraging discussions there.

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Also it was a bit tedious to finish it off. At this point I should like to thank the presence of my friends. Among those I emphasize Sofia, Alastair, Alberto, Angela, Ana and Fina.

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One of the first things that attracted my interest at Warwick was Greg's Lab. I started by turning the knobs of Greg's Box and watching a picture changing on the oscilloscope. I ended up writing a paper with Greg King that became Part II of my thesis.

– Thanks Greg! You told me why and how to do it. You paid attention to me when I needed it. It is nice to feel a bit spoiled.

When I came to Warwick, there was a question in my mind: I want to become a mathematician, but how? Ian Stewart has always kept my self confidence at the correct level, and suddenly I realized that the answer was there.

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Finally I should like to thank Junta Nacional de Investigação Científica e Tecnológica for financial support.

Declaration

This thesis is the original work of the author, with the exception of some sources cited in the text. Part II which has appeared in [32] is joint work of the author with Dr. G.P. King.

Maria Gabriela Miranda Gomes (June 1992)

Summary

Many phenomena in nature can be modeled by differential equations depending on parameters that are being varied continuously. We say that a given solution undergoes a bifurcation with respect to a given parameter if the qualitative behaviour of the system changes arbitrarily close to this solution when the parameter is varied across a critical value. Bifurcation problems can achieve a very high level of complexity because nature is complex. Several assumptions can be made in order to introduce considerable simplifications without going too far from reality. In this thesis we are mainly concerned in setting the problem in a symmetric context and showing that this is a realistic assumption that makes analysis much simpler. We want to emphasize that a lot of behaviour can be much easier to understand and predict when the appropriate symmetry context has been set.

The message in part I of this thesis is that the full set of symmetries is not always obvious. We give examples of phenomena that are modeled by partial differential equations on rectangular domains and show that these problems have more than rectangular symmetry. Such hidden symmetries are found by embedding our problem into a larger one satisfying periodic boundary conditions and then consider all the symmetries that satisfy the original boundary conditions.

In part II we study the behaviour of an electric circuit which can be modeled by a 3-dimensional system of ordinary differential equations. We begin by analysing this system under a symmetry assumption. Then in order to be more realistic we break the symmetry with a small perturbation. Most of the results for the asymmetric system are obtained by numerical and experimental search since a rigorous analysis became much harder. We observe a smooth change in qualitative behaviour by increasing the symmetry breaking perturbation. There is no dramatic change and we conclude that the original symmetry assumption was convenient and not misleading.

Part I

Steady PDEs on Generalized Rectangles

Chapter 1

Introduction

This part deals with a kind of degeneracy that has surprisingly been found in several physical systems. As pointed out by several authors such degeneracies are created by symmetries that are not obvious in bounded domains. This phenomenon was first noticed by Fujii *et al.* [9] and formalized by Armbruster and Dangelmayr [1, 6] for the steady-state reaction-diffusion equation

$$\mathcal{P} \equiv u'' + F(u, \lambda) \;=\; 0 \tag{1.1}$$

where F is a smooth real valued function and $u : \mathbb{R} \to \mathbb{R}$. This PDE is invariant under the euclidean group of translations and reflections. The cited authors are interested in solutions of (1.1) restricted to the interval $[0, \pi]$ satisfying Neumann boundary conditions

$$u'(0) = u'(\pi) = 0.$$

They observe that the only symmetry of the domain is reflection about $\frac{\pi}{2}$

$$\tau : \xi \mapsto \pi - \xi$$

and the behaviour of codimension two bifurcations is not generic in this context. We proceed by describing how this change of genericity occurs. The reason is based on symmetries of $\mathcal{P}(u)$ that do not leave the interval $[0, \pi]$ invariant.

Crawford *et al.* [4] pointed out that this change of genericity may also be present in the simplest codimension one bifurcations. Now, invariance of equation (1.1) under reflection about zero allows the extension of any solution of the Neumann boundary value problem to the interval $[-\pi, \pi]$ by reflection about zero

$$u(-\xi) = u(\xi),$$

and by translational invariance it may be extended 2π-periodically to $\mathbb{R}$. Theorem 1 below shows that this extension method preserves the regularity of the solutions. The proof is based on the Neumann boundary conditions and on the second order structure of equation (1.1). The extended solutions satisfy periodic boundary conditions on the

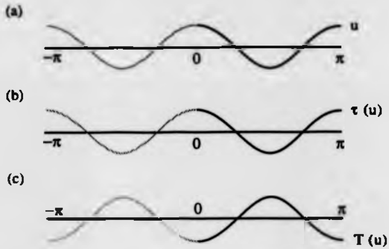


Figure 1.1: (a) Solution satisfying Neumann boundary conditions on $[0,\pi]$ with period π. (b) Its form under the reflection $\tau : \xi \mapsto \pi - \xi$. (c) Its form under the translation $T : \xi \mapsto \xi + \frac{\pi}{2}$.

interval $[-\pi,\pi]$. The reaction-diffusion equation (1.1) on the extended domain has $\mathbf{O}(2)$ as the group of symmetries, generated by the reflection

$$\kappa : \xi \mapsto -\xi$$

and also translations modulo 2π

$$\theta : \xi \mapsto \xi + \theta.$$

Given the translational invariance of equation (1.1) all translations of the extended solution satisfy the periodic boundary value problem. In lemma 1 below we show that any 2π-periodic solution that is invariant under the reflection κ satisfies Neumann boundary conditions at $0, \pi$. On the other hand some of them coincide neither with the original solution nor with its image under the only symmetry τ of the domain. This phenomenon is illustrated in figure 1.1. The original function is invariant under the reflection τ, but it is sent to its negative by reflection about zero followed by translation of $\frac{\pi}{2}$

$$T : \xi \mapsto \xi + \frac{\pi}{2}.$$

We proceed by stating and proving two lemmas that lead directly to the theorem that makes valid the described extension method.

Lemma 1 (lemma 1.1 of Crawford *et al.* [4]) *Solutions to $\mathcal{P}(u) = 0$ satisfying Neumann boundary conditions on $[0,\pi]$ are in 1 : 1 correspondence with κ-invariant solutions satisfying periodic boundary conditions on $[-\pi,\pi]$.*

Proof It is shown above, by construction, that each solution satisfying NBC on $[0,\pi]$ leads to a unique solution satisfying PBC on $[-\pi,\pi]$.

To prove the converse, let $u(\xi)$ be a 2π-periodic solution to $\mathcal{P}(u)=0$ which is invariant under κ. Differentiating the action of κ we get that

$$u'(0)=0 \qquad \text{and} \qquad u'(-\pi)=-u'(\pi).$$

Now 2π-periodicity implies that $u'(-\pi)=u'(\pi)$, so that

$$u'(\pi)=0.$$

Therefore u satisfies Neumann boundary conditions on $[0,\pi]$. □

Lemma 2 below is a particular case of lemma 5.15 of Field *et al.* [8]. It makes a useful observation about smoothness of solutions of the reaction-diffusion equation (1.1).

Lemma 2 *Let $u \in C^1(\mathbb{R})$ be a solution of $\mathcal{P}(u)=0$. Then $u \in C^\infty(\mathbb{R})$.*

Proof Suppose that u is a steady-state solution of the reaction-diffusion equation $\mathcal{P}(u)$. Then

$$u'' + F(u,\lambda) = 0. \tag{1.2}$$

Assume also that $u \in C^1(\mathbb{R})$. This means that u and u' are continuous functions from $\mathbb{R}$ onto $\mathbb{R}$. Now the smoothness of F together with (1.2) imply that u'' is also continuous. Proceeding by induction we get that $u \in C^\infty(\mathbb{R})$. □

Next we state and prove the theorem that validates the extension method described. This a corollary of theorem 5.18 of Field *et al.* [8].

Theorem 1 *Let $\mathcal{P}$ be a reaction-diffusion operator. Then*

1. *Every smooth κ-invariant solution u of $\mathcal{P}$ on $[-\pi,\pi]$ satisfying periodic boundary conditions restricts to a smooth solution of the Neumann problem for $\mathcal{P}$ on $[0,\pi]$.*

2. *Let $u \in C^1([0,\pi])$ be a solution to the Neumann problem for $\mathcal{P}$ on $[0,\pi]$. Then*

   - *u is smooth.*
   - *u extends uniquely to a smooth κ-invariant solution of $\mathcal{P}$ on $[-\pi,\pi]$ satisfying periodic boundary conditions.*

Proof By lemma 1 we know that these statements are true except for the smoothness of the solutions. That is what we concentrate on now.

1. There is nothing to prove here since smoothness of u on $\mathbb{R}$ implies that its restriction to $[0,\pi]$ is also smooth.

2. From the fact that $u \in C^1([0,\pi])$ and the Neumann boundary conditions we have that the extended solution u belongs to $C^1(\mathbb{R})$. By lemma 2, $u \in C^\infty(\mathbb{R})$ and so its restriction to $[0,\pi]$ is also smooth. □

Summarizing, we have that if $\mathcal{P}(u)$ is a reaction-diffusion equation on the line the two following problems are equivalent:

1. $\mathcal{P}(u) = 0$ with Neumann boundary conditions on $[0, \pi]$.
2. Impose O(2)-symmetry to $\mathcal{P}(u) = 0$ on $[-\pi, \pi]$ and then restrict the result to the subspace fixed by the reflection κ.

The only obvious symmetry of problem 1 is the reflection τ about $\frac{\pi}{2}$. This is the only nontrivial symmetry of both the equation and the boundary conditions that leaves the domain $[0, \pi]$ invariant. This reflection can be obtained as a symmetry of problem 2 by reflection about zero composed with a translation of π. However, the converse does not hold. The second problem may have more symmetries that remain hidden in the first.

This change of genericity may occur for more general problems. The ideas described above may be generalized in the following directions:

- Boundary conditions other than Neumann (in particular Dirichlet boundary conditions).
- More complicated operators other than reaction-diffusion, but with similar structure.
- Systems of equation (in particular systems of equations with a structure similar to reaction-diffusion and a mixture of Neumann and Dirichlet boundary conditions on different components).
- More complicated domains in higher dimensions.

The first of these generalizations to Dirichlet boundary conditions was made by Gomes [12] and Crawford *et al.* [4] and there is a brief description of it in chapter 2. The main point is that a periodic extension is allowed also for Dirichlet problems. In this case reflection of solutions across the boundaries must be combined with sign change. A solution u to $\mathcal{P}(u) = 0$ satisfying Dirichlet boundary conditions on $[0, \pi]$ extends to $[-\pi, \pi]$ as

$$u(-\xi) \;=\; -u(\xi).$$

Then we extend periodically on $\mathbb{R}$ as in the Neumann case. Crawford *et al.* [4] also give examples of physical systems that fit into the second generalization, namely the Taylor-Couette and Faraday experiments and the 2-dimensional Bénard convection reproduced in section 1.1 of this introduction. Ashwin [2] applied these methods to the Kuramoto-Sivashinsky equation and we reproduce some of his results in section 1.2.

Field *et al.* [8] define a class of operators more general than the simple reaction-diffusion operator where the extension method still applies. However, the main focus of these authors is the fourth generalization. They present a large class of pairs of manifolds $\Omega \subset \hat{\Omega}$ where extra symmetries obtained from extensions to the larger manifold will change the genericity of the problem on the smaller manifold. They consider $\hat{\Omega}$ as any smooth, compact, connected, Riemannian n-manifold without boundary acted by a group of reflections which divides $\hat{\Omega}$ into several connected components. The smaller manifold with boundary, Ω, is one of these connected components.

After these considerations about how to set PDEs with Neumann boundary conditions into the appropriate symmetry context, we go back to the simple reaction-diffusion equation (1.1). Recall that the group of symmetries of $\mathcal{P}(u) = 0$ with periodic boundary conditions on $[-\pi, \pi]$ is $\mathbf{O}(2)$. Thus there is always a translation-invariant solution which, without loss of generality, we may assume to be $u = 0$. We are interested in studying bifurcations from this trivial solution by varying the parameter λ. Assume that a bifurcation occurs at $\lambda = 0$. By Crawford *et al.* [4] we have the following:

Proposition 1 (proposition 1.2 of Crawford *et al.* [4]) *Under the above hypotheses on $\mathcal{P}(u) = 0$, satisfying Neumann boundary conditions, we have:*

1. *Bifurcating solutions have a well-defined mode number $k \in \mathbb{N}$.*
2. *Generically, when $k > 0$, the bifurcation is a pitchfork.*

Proof By theorem 1 there is a bifurcation at $\lambda = 0$ in equilibrium solutions of $\mathcal{P}(u) = 0$ with periodic boundary conditions on $[-\pi, \pi]$. The group of symmetries of this bifurcation problem is $\mathbf{O}(2)$.

1. Let $L = d\mathcal{P}$ denote the linearized equations about $u = 0$ at $\lambda = 0$. By $\mathbf{O}(2)$ symmetry we expect ker L to be either one or two dimensional, since, by Golubitsky *et al.* [11] p.330, irreducible representations of $\mathbf{O}(2)$ have those dimensions. We may write the action of $\mathbf{SO}(2)$ on ker L as
$$\theta : z \mapsto e^{ik\theta} z, \tag{1.3}$$
where $k = 0$ in the simple eigenvalue case and $k > 0$ in the double eigenvalue case. The integer k defined in (1.3) is the *mode number*. If $k > 0$ the bifurcating solutions are $\frac{2\pi}{k}$-periodic.

2. From part 1 of this proof we see that in the periodic boundary conditions problem ker L is spanned by the eigenfuntion $e^{ik\xi}$. We denote this eigenspace by
$$\tilde{V}_k = \operatorname{span}\{e^{ik\xi}\}.$$
The action of $\mathbf{O}(2)$ on the eigenfunction is generated by $\mathbf{SO}(2)$ acting as (1.3) together with the reflection
$$\kappa : z \mapsto \bar{z}.$$
The solutions to $\mathcal{P}(u) = 0$ satisfying Neumann boundary conditions are found in
$$\mathrm{Fix}(\kappa) = \{z \in \ker L | z = \bar{z}\} = \mathbb{R}.$$
Thus when Neumann boundary conditions are imposed, the eigenspace of L is the subspace of $\tilde{V}_k$ that is fixed by the action of κ. This eigenspace is denoted by
$$V_k = \operatorname{span}\{\cos(k\xi)\}.$$
If $k > 0$ this fixed point subspace is invariant under the translation of half a period, $\frac{\pi}{k}$, which sends the eigenfunction to its negative. Thus, all bifurcations are pitchforks. □

There are two relevant consequences of proposition 1. The first is the pattern formation associated with the existence of a mode number. This phenomenon has been observed in experiments and it would not be expected if the periodic extension had not been taken into account. The second consequence is the existence of a $\mathbf{Z}_2$-symmetry in the problem of bifurcating solutions with even mode number, in which case the symmetry of the domain acts trivially on the solution, as illustrated in figure 1.1 for $k = 2$. Note that if the mode number is odd the solution is sent to its negative by the symmetry of the domain. So the extension method does not introduce any new reflection symmetry in this case.

In section 2.3 we prove an analogue of proposition 1 for reaction-diffusion equations satisfying Neumann boundary conditions on n-dimensional rectangles. In this case the group acting on the extended problem with periodic boundary conditions is $\mathbf{O}(2)^n$. This implies pattern formation in the n directions that generate the rectangle. So each solution has n associated mode numbers. By analogy we show that all steady-state bifurcations from the trivial solution $u = 0$ are pitchforks. By considering only the symmetries of the domain, pitchfork bifurcations should not be expected if all the mode numbers are even. So also here there is a change of genericity by considering the periodic extension.

Armbruster and Dangelmayr [1, 6] use these ideas to study steady-state mode interactions in reaction-diffusion equations on the interval with Neumann boundary conditions. Assume that $\mathcal{P}(u)$ depends on an additional parameter r. Associated to each mode number there is a curve along which the linearized Neumann boundary conditions problem has a single eigenvalue. Generically, two such curves meet at some point. Suppose that the curves corresponding to mode numbers $k, l \in \mathbf{N}$ meet when the parameters are set to zero. This implies that L has a double zero eigenvalue and no other eigenvalues on the imaginary axis.

Assume that $k \neq l$. When extended to periodic boundary conditions, ker L is the direct sum of two irreducible representations of the group $\mathbf{O}(2)$ as

$$\hat{V}_k \oplus \hat{V}_l,$$

where $\hat{V}_k$ and $\hat{V}_l$ are defined in the proof of proposition 1. The action of $\mathbf{O}(2)$ on such eigenspace is generated by

$$\begin{aligned} \theta &: (z, w) \longmapsto (e^{ik\theta}z, e^{il\theta}w) \\ \kappa &: (z, w) \longmapsto (\bar{z}, \bar{w}) \end{aligned} \tag{1.4}$$

We may assume without loss of generality that k and l are coprime. Otherwise we factor out the kernel of the group action and restore it when interpreting the results.

By fixing the parameter $r = 0$ and keeping λ as the bifurcation parameter, after a Liapunov-Schmidt reduction we get the bifurcation equations

$$f(z, w, \lambda) \;=\; 0,$$

where $f : \hat{V}_k \oplus \hat{V}_l \times \mathbf{R} \to \hat{V}_k \oplus \hat{V}_l$. We know that f is $\mathbf{O}(2)$-equivariant under the action (1.4). Since $\mathrm{Fix}(\kappa) = \mathbf{R}^2$ the bifurcation equations corresponding to Neumann boundary conditions are just

$$g(x, y, \lambda) \;=\; 0,$$

where $x = \mathrm{Re}(z)$, $y = \mathrm{Re}(w)$ and $g = f|\mathbb{R}^2 \times \mathbb{R} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. By invariant theory methods Armbruster and Dangelmayr [1, 6] compute the generators for the invariant functions and equivariant mappings under the action (1.4). Then they restrict the result to $\mathrm{Fix}(\kappa)$. The result is as follows:

Theorem 2 (Armbruster and Dangelmayr [1, 6]) *Assume that $\mathcal{P}(u) = 0$ undergoes a simultaneous bifurcation of the two modes $k, l \in \mathbb{N}$ from $u = 0$ when the parameter λ crosses zero. Then the projection of the bifurcation equations onto $\ker L$ are of the form*

$$\begin{aligned} ax + cx^{l-1}y^k &= 0 \\ by + dx^l y^{k-1} &= 0, \end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ.

In reference [1] these authors give the classification of the this system up to codimension two and the corresponding bifurcation diagrams. Their results say thay if k and l are large enough the system behaves basically as a generic $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-invariant problem with a slight bending of branches in the bifurcation diagram. By looking at the equations in theorem 2 this behaviour is not surprising: the first term in each line is what would be expected from $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry and the second two terms break this at high order. What is surprising is the form of the system. The Neumann boundary conditions should lead to a generic system with $\mathbf{Z}_2$-symmetry. However, the existence of an extended problem with greater symmetry causes a degeneracy that may change the result dramatically.

Gomes [12] proved the analogue of theorem 2 for two dimensional rectangles. Furthermore, in section 2.4 we show that it generalizes for n dimensions. The proof uses combinatorial arguments based on representation theory and invariant theory. This is the central result of this thesis. It applies to the steady-state reaction-diffusion equation

$$\mathcal{P} \equiv \Delta u + F(u, \lambda) = 0$$

where F is a smooth real valued function and $u : \mathbb{R}^n \to \mathbb{R}$. We are interested in solutions $u(\xi)$ to $\mathcal{P}(u) = 0$ on the generalized rectangle $[0, \ell_1] \times \cdots \times [0, \ell_n]$, where all the ℓ_j are different, with Neumann boundary conditions

$$\frac{\partial u}{\partial \xi_i}(\xi) = 0 \qquad \text{when} \qquad \xi_i = 0, \ell_i \qquad \text{for} \qquad 1 \le i \le n.$$

We assume that $\mathcal{P}(u) = 0$ undergoes a simultaneous bifurcation of the two modes $(k_1, \ldots, k_n), (l_1, \ldots, l_n) \in \mathbb{N}^n$ from $u = 0$ when the parameter λ crosses zero. Then the form of the projection of the bifurcation equations onto $\ker L$ depends on the mode numbers as follows:

1. If all k_j have the same parity and all l_j have the same parity we have

$$\begin{aligned} ax + bx^{l-1}y^k &= 0 \\ cy + dx^l y^{k-1} &= 0, \end{aligned}$$

   where $k = \max_j k_j$ and $l = \max_j l_j$.

2. Otherwise

$$\begin{aligned} ax &= 0 \\ cy &= 0, \end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ. In case 1 the bifurcation equations reduce to those of Armbruster and Dangelmayr when $n = 1$. However, their assumption that k and l are coprime is no longer always valid. In case 2 the equations are $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant and those have been studied by Golubitsky and Schaeffer [30]. In this case there are no degeneracies created by the periodic extention.

Recall that in this description of problems on generalized rectangles we have assumed that all the edges have different lengths. This is the main concern of chapter 2. Apart from this, some partial results on simultaneous bifurcations of an arbitrary number of modes are also given. We say partial results because a general form for the bifurcation equations cannot be given: the generators of the restricted invariants and equivariants must be computed algorithmically.

From now on we consider the effect of equal edges. Suppose that the domain is an n-dimensional rectangle $[0, \ell_1] \times \ldots \times [0, \ell_n]$ with $m \leq n$ equal edges

$$\ell_1 = \cdots = \ell_m$$

and all the other ℓ_j are different. Then given the euclidean invariance of the reaction-diffusion equation $\mathcal{P}(u)$, the symmetry group of the periodic boundary conditions problem is the semi-direct sum $\mathbf{O}(2)^n \dot{+} \mathbf{S}_m$ where $\mathbf{S}_m$ is the group of permutations of m elements.

Chapter 3 deals with the case where all the edges are equal

$$\ell_1 = \cdots = \ell_n$$

and our generalized rectangle is actually a generalized cube. The symmetry group of the periodic boundary conditions problem is now $\mathbf{O}(2)^n \dot{+} \mathbf{S}_n$ This addition of the group of permutations makes the problem extremely complicated. All results depend very strongly on n. For a single mode bifurcation with Neumann boundary conditions on such a domain we expect ker L to have dimension $n!$ and the difficulty of analysing the bifurcation equations grows very fast with n. For $n = 1$ the domain is just an interval, which was studied before. A description of the case $n = 2$ was given by Crawford [3]. If $n = 3$ the problem is already too complicated. As we will show, the symmetries introduced by the periodic extension depend on the mode numbers in a very complicated way.

In chapter 4 some results for 2-mode interactions on squares and 3-mode interactions on rectangles apply to the 3-dimensional Bénard convection problem. For such mode interactions, the kernel of the linearized operator is expected to have dimensions three or four. Depending on the mode numbers we may or not give a general form for the bifurcation equations. In some cases the generators of the invariants must be computed algorithmically, which is an obstacle to obtaining of general results. A Liapunov-Schmidt reduction will be performed near a 3-mode interaction point on a box with a rectangular horizontal section and a 2-mode interaction point on a box with a square horizontal section.

In appendices A and B we analyse normal forms with $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $\mathbf{Z}_2 \oplus \mathbf{D}_4$ symmetry. Tables for existence and criticality of branches are given. If we have a set of bifurcation equations with such symmetries, the bifurcation diagrams can be drawn directly from the tables in the appendices. The use of the appendices is illustrated in chapter 4 for sets of equations obtained by Liapunov-Schmidt reduction.

The aim of the following two sections is to illustrate how the abstract ideas described above apply to concrete physical problems. Rather than trying to obtain new results for these particular problems, we want to show how our methods can lead to new results.

1.1 Two Dimensional Bénard Convection

Apart from having an interest on its own, the Bénard convection problem is often chosen to illustrate in a concrete setting many of the abstract ideas of bifurcation theory. Also it gives a good intuition for other generalizations. This problem can be taken up from different points of view. An assumption often made is that the fluid is confined to an infinite layer bounded by two horizontal planes. Since here we are mainly interested in properties induced by the boundary conditions, we refer to Kidachi [15] and Metzener [16] for work on finite domains. We take a different approach here. We want to use all the symmetry information that we can get. The methods used here apply in a much more general setting as described above in this introduction.

We proceed by giving a brief description of the Bénard problem in the 2-dimensional rectangle $\Omega = [0, \pi\ell_1] \times [0, \pi\ell_2]$. Our main interest is to show how the extension method applies in this case. This method tells us what are the actual symmetries of the problem and how to set a generic problem satisfying these symmetry constraints. Then it can be shown that this abstract setting is isomorphic to the Bénard problem and it has the advantages of being much simpler and representing a large class of systems. This section is also a motivation for chapter 4 where we analyse in more detail the same Bénard problem in a 3-dimensional box.

As in Golubitsky *et al.* [11], the Boussinesq approximation of the equations for the 2-dimensional Bénard convection may be written as

$$\begin{aligned}
&\frac{1}{\sigma}v.\nabla v_1 = \Delta v_1 - \frac{\partial p}{\partial \xi_1} \\
&\frac{1}{\sigma}v.\nabla v_2 = \Delta v_2 - \frac{\partial p}{\partial \xi_2} + \Theta \\
&v.\nabla\Theta = \Delta\Theta + Rv_2 \\
&\frac{\partial v_1}{\partial \xi_1} + \frac{\partial v_2}{\partial \xi_2} = 0
\end{aligned} \tag{1.5}$$

where $v = (v_1, v_2)$ is the velocity vector, Θ describes the deviation of temperature, p is the pressure and the parameters R and σ are, respectively, the Rayleigh number and the Prandtl number. The boundary conditions are a mixture of Neumann and

Dirichlet as

$$v_1 = \frac{\partial v_2}{\partial \xi_1} = \Theta_{\xi_1} = 0 \qquad \text{for} \qquad \xi_1 = 0, \pi\ell_1 \tag{1.6}$$
$$\frac{\partial v_1}{\partial \xi_2} = v_2 = \Theta = 0 \qquad \text{for} \qquad \xi_2 = 0, \pi\ell_2.$$

In order to satisfy the last equation of (1.5) we express the velocity in terms of the stream function Ψ

$$v_1 = \frac{\partial \Psi}{\partial \xi_2} \qquad v_2 = -\frac{\partial \Psi}{\partial \xi_1}.$$

The pressure can be eliminated by subtracting $\frac{\partial}{\partial \xi_1}$ of the second equation from $\frac{\partial}{\partial \xi_2}$ of the first. This operation together with the scaling

$$\xi_1 \mapsto \frac{1}{\ell_1}\xi_1 \qquad \xi_2 \mapsto \frac{1}{\ell_2}\xi_2 \qquad \Theta \mapsto \ell_2^3\Theta \qquad R \mapsto \ell_2 R$$

yields the system

$$\begin{aligned} \Delta_r^2\Psi - \frac{1}{r}\frac{\partial \Theta}{\partial \xi_1} + \frac{1}{\sigma}J_r(\Psi, \Delta\Psi) &= 0 \\ \Delta_r\Theta - \frac{R}{r}\frac{\partial \Psi}{\partial \xi_1} + J_r(\Psi, \Theta) &= 0, \end{aligned} \tag{1.7}$$

where the parameter

$$r = \frac{\ell_1}{\ell_2}$$

has been introduced in the equations and some operators have been scaled as

$$\begin{aligned} \Delta_r &= \frac{1}{r^2}\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \\ \nabla_r &= \left(\frac{1}{r}\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right) \\ J_r(f,g) &= \frac{1}{r}\left(f_{\xi_1}g_{\xi_2} - f_{\xi_2}g_{\xi_1}\right). \end{aligned}$$

The domain has been scaled to $\Omega = [0,\pi] \times [0,\pi]$ and the boundary conditions are

$$\Psi = \Psi_{\xi_1\xi_1} = \Theta_{\xi_1} = 0 \qquad \text{for} \qquad \xi_1 = 0, \pi \tag{1.8}$$
$$\Psi = \Psi_{\xi_2\xi_2} = \Theta = 0 \qquad \text{for} \qquad \xi_2 = 0, \pi.$$

Let $u = (\Psi, \Theta)$ be a smooth solution to Φ satisfying the boundary conditions (1.8) on Ω. We define an extension $\hat{u}$ by reflecting each component of u across the boundaries of the domain as

$$\kappa_1 : (\Psi,\Theta)(\xi_1,\xi_2) \mapsto (-\Psi,\Theta)(-\xi_1,\xi_2)$$
$$\kappa_2 : (\Psi,\Theta)(\xi_1,\xi_2) \mapsto (-\Psi,-\Theta)(\xi_1,-\xi_2)$$

and extend periodically to the whole $\mathbb{R}^2$. By construction, the extended function $\hat{u}$ is K-invariant where K is the group generated by κ_1, κ_2, which is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. It can be shown that $\hat{u}$ is a smooth solution of the same equations on the larger domain $\hat{\Omega} = [-\pi, \pi] \times [-\pi, \pi]$ satisfying periodic boundary conditions. The opposite is also true: if $\hat{u}$ is a smooth K-invariant solution to Φ satisfying periodic boundary conditions on $\hat{\Omega}$ then its restriction to Ω is a smooth solution of the same equations satisfying the boundary conditions (1.8). These results are proved in section 4.3 for the Bénard problem in 3-dimensional domains.

We observe that equations (1.7) admit the translation invariant solution $u = 0$. We are interested in steady states bifurcating from this trivial branch when the Rayleigh number R is increased. Let L_r denote the linearization of the operator (1.7) around $u = 0$. In order to locate bifurcation points in the parameter space and compute the corresponding eigenfunctions we solve the one parameter family of equations

$$L_r(u, R) = 0. \tag{1.9}$$

As explained before, bifurcating solutions have a well defined set of mode numbers $(k_1, k_2) \in \mathbb{N}^2$ induced by the periodic extension. By imposing the boundary conditions (1.8) we get the eigenmode u_k with components

$$\begin{aligned} \Psi &= \frac{k_1}{r} \sin(k_1\xi_1)\sin(k_1\xi_2) \\ \Theta &= -\left(\frac{k_1^2}{r^2} + k_2^2\right)\cos(k_1\xi_1)\sin(k_1\xi_2) \end{aligned}$$

when the Rayleigh number is at the critical value

$$R_k = \frac{\left(\frac{k_1^2}{r^2} + k_2^2\right)^3}{\frac{k_1^2}{r^2}}.$$

By plotting the critical Rayleigh numbers in the (r, R) parameter space we get the curves in figure 1.2. By fixing the unfolding parameter r, we have a bifurcation from $u = 0$ when the bifurcation parameter R is increased across some of these curves. It is well known (see Golubitsky *et al.* [11]) that the qualitative behaviour of system (1.7) is preserved by projection onto ker L_r. This process is the so called Liapunov-Schmidt reduction. Under the assumption of periodic boundary conditions, ker L_r must be an irreducible representation of $K \dot{+} \mathbf{T}^2$ at a single mode bifurcation point, where K acts as before and $\mathbf{T}^2$ acts as translations of the domain. The effect of imposing the boundary conditions (1.8) is to restrict this representation to $\mathrm{Fix}(K)$. It can be shown that in this case ker L_r is isomorphic to a representation defined before in the context of reaction-diffusion equations with Neumann boundary conditions (see chapter 4 for an explicit isomorphism in 3-dimensional domains). So we can apply directly here the results stated before in this introduction, but the interpretation may differ because the isomorphism is not the identity.

For a simultaneous bifurcation of two distinct modes, ker L_r is the direct sum of two representations as above. A generalization of results applicable to such situations

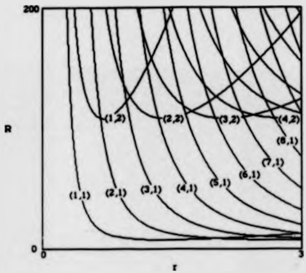


Figure 1.2: Location of bifurcations in the (r, R) space.

is stated above in this introduction and proved in chapter 2. We reproduce here the restriction of the final result to the 2-dimensional case for ease of exposition.

Denote the bifurcating modes by $(k_1, k_2), (l_1, l_2) \in \mathbb{N}^2$ and $x, y \in \mathbb{R}$ the respective amplitudes. Then the form of the projection of the bifurcation equations onto $\ker L_r$ depends on the mode numbers as follows:

1. If k_1, k_2 have the same parity and l_1, l_2 have the same parity we have

$$\begin{aligned} ax + bx^{l-1}y^k &= 0 \\ cy + dx^l y^{k-1} &= 0, \end{aligned}$$

   where $k = \max(k_1, k_2)$ and $l = \max(l_1, l_2)$.

2. Otherwise

$$\begin{aligned} ax &= 0 \\ cy &= 0, \end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ.

On the other hand the symmetries induced by the two reflections that leave the domain invariant are

1. $\mathbf{Z}_2$ if k_1, k_2 have the same parity and l_1, l_2 have the same parity.

2. $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ otherwise.

In case 2 the bifurcation equations are expected to be a generic $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant with or without the extension method. The difference comes in case 1. Without the extension method we should expect the bifurcation equations to be a generic $\mathbf{Z}_2$-equivariant. As a consequence of extending the problem we find more symmetries on the nonlinear part. We can tell which coefficients are expected to vanish and at which order we can truncate the bifurcation equations without misleading the result.

Bifurcations more often considered in the literature are the ones occuring at onset. From figure 1.2 we see that onset mode interactions have mode numbers of the form $(n,1)-(n+1,1)$ and these do not satisfy the parity conditions in case 1. Therefore the bifurcation equations for onset mode interactions in the 2-dimensional Bénard problem have always $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry and the extension method is not needed to make this prediction (see Metzener [16]). It would be relevant for other mode interactions found in figure 1.2. Some examples are

- (2,2)-(3,1)
- (2,2)-(5,1)
- (4,2)-(7,1)

among many others satisfying the conditions in case 1. Further analysis is not performed here. Recall that the point we wanted to make is that the extension method applies in the Bénard convection problem and in some cases (depending on the mode numbers) it is really worth applying. It can save the work of computing coefficients in the bifurcation equations that can be predicted to be zero and it helps to decide where the equations can be truncated.

1.2 Kuramoto-Sivashinsky Equation

The results summarized in this introduction have been applied by Ashwin [2] to the Kuramoto-Sivashinsky equation describing the motion of a flame front. He examines steady solutions on a rectangular domain with Neumann boundary conditions. There is a double interest in reproducing some of his results here

- As a physical application, this work is a motivation for the methods developed in this thesis.
- The particular setting of this problem makes it possible to obtain further results. Many times in this thesis we refer to an algorithm that does part of the work. Beyond this it seems that no more general results can be obtained.

We proceed by stating the problem and summarizing the results. Some notations will be changed for uniformity with the ones we have previously adopted. By taking the limit of zero gas expansion the steady equations can be written as

$$\mathcal{P}(u) \quad = \quad \Delta^2 u + \Delta u + (\nabla u)^2 = 0. \qquad (1.10)$$

The domain is denoted by $\Omega = [0, a\pi] \times [0, b\pi]$ and the Neumann boundary conditions are

$$\begin{aligned} \frac{\partial u}{\partial \xi_1} = \frac{\partial \Delta u}{\partial \xi_1} = 0 &\quad \text{for} \quad \xi_1 = 0, a\pi \\ \frac{\partial u}{\partial \xi_2} = \frac{\partial \Delta u}{\partial \xi_2} = 0 &\quad \text{for} \quad \xi_2 = 0, b\pi. \end{aligned}$$

Let the function u be a solution of equation (1.10) satisfying Neumann boundary conditions. By reflecting u across the boundaries of the domain we get a new function $\hat{u}$ such that

$$\begin{aligned} \hat{u}(-\xi_1, \xi_2) &= \hat{u}(\xi_1, \xi_2) \\ \hat{u}(\xi_1, -\xi_2) &= \hat{u}(\xi_1, \xi_2). \end{aligned} \tag{1.11}$$

Then we extend $\hat{u}$ periodically to the whole $\mathbb{R}^2$. Note that equation (1.10) is invariant under an action of the euclidean group $E(2)$. Thus the extended function $\hat{u}$ is a solution of this equation satisfying periodic boundary conditions on the larger domain $\tilde{\Omega} = [-a\pi, a\pi] \times [-b\pi, b\pi]$. It may also be shown that this extension preserves the regularity of the solution. The reverse is also true. Thus, solutions of (1.10) on Ω satisfying Neumann boundary conditions are in 1 : 1 correspondence with solutions of the same equation satisfying periodic boundary conditions on $\tilde{\Omega}$ and the invariance (1.11).

Note that $u = 0$ is a translation-invariant solution of equation (1.10). Ashwin examines steady-state bifurcations by regarding the two numbers (a, b) as bifurcation parameters. Denote $L = d\mathcal{P}$ the linearization of $\mathcal{P}$ about $u = 0$. By Golubitsky and Schaeffer [30], there is a bifurcation when L has a nontrivial kernel.

The periodic extension property implies the eigenvectors can be written as

$$u_k = \cos\left(\frac{k_1 \xi_1}{a}\right) \cos\left(\frac{k_2 \xi_2}{b}\right),$$

where $(k_1, k_2) \in \mathbb{N}^2$ are called mode numbers. The set of bifurcation points in the (a, b) plane is shown in figure 1.3, where each line corresponds to fixed mode numbers (k_1, k_2).

Bifurcation diagrams in a neighbourhood of some mode interaction points are drawn in Ashwin [2]. Using computer algebra, he projects equation (1.10) onto ker L, which is a finite dimensional space. Then a bifurcation analysis is performed on these equations. The method used for the projection is the so called Liapunov-Schmidt reduction as in Golubitsky and Schaeffer [30].

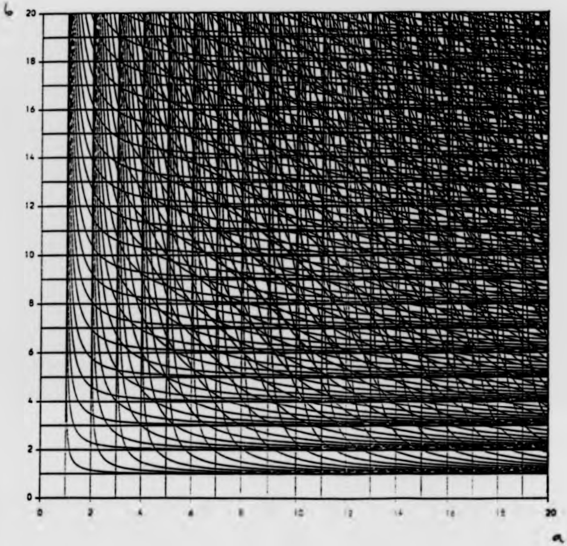


Figure 1.3: Location of bifurcations in the (a, b) plane. Taken from Ashwin [2].

Chapter 2

Boundary Conditions and Symmetry

2.1 Introduction

The aim of this chapter is to generalize, in a very natural way, the results obtained by Armbruster and Dangelmayr [1, 6] and Fujii *et al.* [9] for reaction-diffusion equations with Neumann boundary conditions (NBC) on the interval to n-dimensional rectangles. We give a brief description of how to obtain analogous results when Dirichlet boundary conditions (DBC) are imposed. This case is not explored in much detail here because after setting the symmetry context where such problems are generic, which we do, a complete analysis follows as in the NBC case. These results also apply to more general PDEs but we do not explore this direction here.

We proceed by describing the problem that we will concentrate on and summarizing the results obtained. Let $\mathcal{P}(u)$ denote a reaction-diffusion equation on an n-dimensional rectangle with Neumann or Dirichlet boundary conditions. We are interested in the solutions to $\mathcal{P}(u) = 0$. In section 2.2 we give the appropriate symmetry context, where this problem is generic. The basic idea in the NBC case is to state a periodic boundary conditions (PBC) problem using the symmetry $\mathbf{O}(2)^n$, where the superscript means cartesian product, and then restrict the result to the subspace fixed by the reflections across the boundaries of the domain. The main point of Armbruster and Dangelmayr [1, 6] and Fujii *et al.* [9] and emphasized by Crawford *et al.* [4] is that in the case $n = 1$ this procedure leads to a problem with more symmetry than we would expect by considering only the shape of the domain. This was explained with more detail in chapter 1. In the DBC case we also construct a periodic extension and then restrict the result to the subspace fixed by reflections across the boundaries combined with sign change. This extension requires an extra $\mathbf{Z}_2$-symmetry of the operator $\mathcal{P}$.

Further results in this chapter will be obtained for Neumann problems only since this is the simplest case where the extension method applies. Analysis of other boundary conditions (in particular, Dirichlet) can be carried out in a very similar way. In section 2.3 we assume that the differential operator $\mathcal{P}$ is parametrized by λ and $\mathcal{P}(u) = 0$ has a trivial solution $u = 0$, which undergoes a steady-state bifurcation at $\lambda = 0$. Let $L = d\mathcal{P}$ denote the linearization about $u = 0$ at $\lambda = 0$. We are interested in the projection of solutions of $\mathcal{P}(u) = 0$ onto ker L for small λ. We show that ker L is one dimensional and the resulting normal form is a generic $\mathbf{Z}_2$-invariant system. We remark that this result is completly general in the sense that it does not depend either on the dimension n or on the mode numbers $k \in \mathbb{N}^n$. Thus any bifurcation from the trivial solution is a pitchfork. Apart from having an interest on its own, this section also introduces an irreducible representation of $\mathbf{O}(2)^n$ which gives the basic blocks for the construction of more complicated representations of groups in later sections.

Section 2.4 is concerned with the case where the operator $\mathcal{P}$ depends on an additional parameter r. In this case we expect the existence of values of r such that two modes bifurcate simultaneously from the trivial solution when we vary the bifurcation parameter λ. In this section we state and prove the main theorem which describes the generic bifurcations of a system near a mode interaction. Now ker L is two dimensional and the normal form is still independent of n but depends on the mode numbers $k, l \in \mathbb{N}^n$.

In section 2.5 we assume that $\mathcal{P}$ depends on $m-1$ parameters apart from the bifurcation parameter λ. In this case the simultaneous bifurcation of m distinct modes is generic. The defining conditions for the form of the restricted bifurcation equations are given but we do not write the equations explicit. The reason for this omission is that the computation of a minimal set of generators of the invariants and equivariants involves very complicated combinatorics when more than two modes bifurcate simultaneously. However, we leave the problem in a state that will permit an algorithm to be applied.

2.2 The Appropriate Symmetry Context

Let $\mathcal{P}(u)$ denote a reaction-diffusion equation on $\mathbb{R}^n$

$$\mathcal{P} \equiv \Delta u + F(u, \lambda) = 0 \tag{2.1}$$

where F is a smooth real valued function and $u : \mathbb{R}^n \to \mathbb{R}$. This equation is invariant under translations and reflections. We proceed by describing the symmetric structure that the boundary conditions impose on the solution set of (2.1). Sections 2.2.1 and 2.2.2 deal with, respectively, Neumann and Dirichlet boundary conditions.

2.2.1 Neumann Boundary Conditions

Let $u(\xi)$ be a solution of $\mathcal{P}(u) = 0$ on the generalized rectangle $\Omega = [0, \pi\ell_1] \times \cdots \times [0, \pi\ell_n]$, where all ℓ_j are different. Assume that $u(\xi)$ satisfies the NBC

$$\frac{\partial u}{\partial \xi_i}(\xi) = 0 \qquad \text{when} \qquad \xi_i = 0, \pi\ell_i \qquad \text{for} \qquad 1 \leq i \leq n. \tag{2.2}$$

Then u may be extended to a solution of the same PDE on the whole $\mathbb{R}^n$ that satisfies PBC on the larger rectangle $\hat{\Omega} = [-\pi\ell_1, \pi\ell_1] \times \cdots \times [-\pi\ell_n, \pi\ell_n]$. We make this extension by reflection across the boundaries

$$\hat{u}(\kappa\xi) = \hat{u}(\xi). \tag{2.3}$$

where κ belongs to the group K generated by

$$\kappa_j : \xi_j \mapsto -\xi_j \qquad \text{for} \qquad 1 \leq j \leq n,$$

which is isomorphic to $(\mathbf{Z}_2)^n$. Note that, by construction, the solution $\hat{u}$ that we have just defined on $\hat{\Omega}$ is K-invariant.

In lemma 3 we show that the converse is also true. More precisely, suppose that $\hat{u}$ is a solution to the reaction-diffusion equation (2.1) on $\hat{\Omega}$ that is K-invariant. Then the restriction u to Ω satisfies the NBC.

Now we extend $\hat{u}$ periodically to the whole $\mathbb{R}^n$. Denote $\pi\ell = (\pi\ell_1, \ldots, \pi\ell_n)$. We say that a solution u is $2\pi\ell$-periodic if and only if it is $2\pi\ell_j$-periodic along the direction ξ_j, for $1 \leq j \leq n$. The solution that we have constructed is $2\pi\ell$-periodic. Note that the two statements

- u is K-invariant and $2\pi\ell$-periodic on $\mathbb{R}^n$
- u is K-invariant and satisfies periodic boundary conditions on $\hat{\Omega}$

are equivalent. From now on we use either of them arbitrarily.

In theorem 3 we show that the extension procedure described above preserves the regularity of the solutions.

Lemma 3 *Solutions to $\mathcal{P}(u) = 0$ satisfying Neumann boundary conditions on Ω are in 1 : 1 correspondence with K-invariant solutions satisfying periodic boundary conditions on $\hat{\Omega}$.*

Proof This proof is very similar to the corresponding argument in one dimensional domains, lemma 1 in chapter 1. It is shown above, by construction, that each solution satisfying NBC on Ω leads to a unique solution satisfying PBC on $\hat{\Omega}$.

To prove the converse, let $\hat{u}(\xi)$ be a $2\pi\ell$-periodic solution to $\mathcal{P}(u) = 0$ which is invariant under K. Differentiating the action of each of the generators of K in the direction that is being reflected we get that

$$\frac{\partial u}{\partial \xi_j}(0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial \xi_j}(-\pi\ell_j) = -\frac{\partial u}{\partial \xi_j}(\pi\ell_j) \qquad \text{for} \qquad 1 \leq j \leq n.$$

Now $2\pi\ell$-periodicity implies that

$$\frac{\partial u}{\partial \xi_j}(-\pi\ell_j) = \frac{\partial u}{\partial \xi_j}(\pi\ell_j) \qquad \text{for} \qquad 1 \leq j \leq n,$$

so that

$$\frac{\partial u}{\partial \xi_j}(\pi\ell_j) = 0 \qquad \text{for} \qquad 1 \leq j \leq n.$$

Therefore the restriction u to Ω satisfies Neumann boundary conditions. □

It is necessary here to discuss the regularity of solutions u of the steady reaction-diffusion equation $\mathcal{P}(u) = 0$. Lemma 4 and theorem 3 below are corollaries of lemma 5.15 and theorem 5.18 of Field *et al.* [8]. These authors consider more general operators as well as more general domains. We state these simpler versions here for completeness and ease of exposition.

Lemma 4 *Assume that $\mathcal{P}$ is a reaction-diffusion operator (2.1). Let $u \in C^1(\mathbb{R}^n)$ be a solution of $\mathcal{P}$. Then $u \in C^\infty(\mathbb{R}^n)$.*

Proof This follows from the ellipticity of $\mathcal{P}$ with respect to u (see Field *et al.* [8]). □

From lemmas 3 and 4 we immediately obtain the following:

Theorem 3 *Let $\mathcal{P}$ be a K-invariant reaction-diffusion operator. Then*

1. *Every smooth K-invariant solution $\hat{u}$ of $\mathcal{P}$ on $\hat{\Omega}$ satisfying periodic boundary conditions restricts to a smooth solution of the Neumann problem for $\mathcal{P}$ on Ω.*
2. *Let $u \in C^1(\Omega)$ be a solution to the Neumann problem for $\mathcal{P}$ on Ω. Then*
   - *u is smooth.*
   - *u extends uniquely to a smooth K-invariant solution of $\mathcal{P}$ on $\hat{\Omega}$ satisfying periodic boundary conditions.*

Proof By lemma 3 we know that these statements are true except for the smoothness of the solutions. That is what we concentrate on now.

1. There is nothing to prove here since smoothness of u on $\hat{\Omega}$ implies that its restriction to Ω is also smooth.
2. From the fact that $u \in C^1(\Omega)$ and the Neumann boundary conditions we have that the extended solution $\hat{u}$ belongs to $C^1(\mathbb{R}^n)$. By lemma 4, $u \in C^\infty(\mathbb{R}^n)$ and so its restriction to Ω is also smooth.

□

Now the PBC problem has $\mathbf{O}(2)^n$-symmetry, where the superscript means cartesian product. This group may also be written as $K \dot{+} \mathbf{T}^n$ where K acts by reflecting the components of ξ as before and $\mathbf{T}^n$ is generated by translations as

$$\theta_j : \xi_j \mapsto \xi_j + \theta_j \qquad \text{for} \qquad 1 \le j \le n,$$

The appropriate way of studying the NBC problem is to study first the PBC problem using the $\mathbf{O}(2)^n$-symmetry, and then restrict the results to the fixed point subspace $\mathrm{Fix}(K)$.

On the other hand the only obvious symmetries of the Neumann problem, if the extension property is not taken into account, are the symmetries of the domain. In the case of rectangular domains these symmetries form the group $(\mathbf{Z}_2)^n$ generated by

$$\tau_j : \xi_j \mapsto \pi\ell_j - \xi_j \qquad \text{for} \qquad 1 \le j \le n.$$

These symmetries are included in the ones obtained by the extension procedure but the converse may not hold (see chapter 1 for an example in the 1-dimensional case). This means that there is a change in generic behaviour if we apply one method instead of the other to solve the problem. There are symmetry constraints being ignored if we take only the symmetries of the domain into account. The aim of later sections is to show that this change of genericity occurs for rectangles of any dimension and also to describe explicitly what is the generic behaviour for single mode bifurcations and mode interactions.

2.2.2 Dirichlet Boundary Conditions

The situation for DBC is slightly more complicated but the basic ideas are the same. In this section we describe the modifications from the previous one.

Recall that $\mathcal{P}(u)$ denotes a reaction-diffusion operator on $\mathbb{R}^n$ as (2.1). Let $u(\xi)$ be a solution of $\mathcal{P}(u) = 0$ on the generalized rectangle $\Omega = [0, \pi\ell_1] \times \cdots \times [0, \pi\ell_n]$, where all ℓ_j are different. Assume that $u(\xi)$ satisfies the DBC

$$u(\xi) = 0 \qquad \text{when} \qquad \xi_i = 0, \pi\ell_i \qquad \text{for} \qquad 1 \leq i \leq n. \tag{2.4}$$

Then u may be extended to a function on the whole $\mathbb{R}^n$ that satisfies PBC on the larger rectangle $\hat{\Omega} = [-\pi\ell_1, \pi\ell_1] \times \cdots \times [-\pi\ell_n, \pi\ell_n]$. We make this extension by reflection across the boundaries composed with sign change

$$\hat{u}(\kappa_j\xi) = -\hat{u}(\xi) \qquad \text{for} \qquad 1 \leq i \leq n. \tag{2.5}$$

where the κ_j act on $\mathbb{R}^n$ as

$$\kappa_j : \xi_j \mapsto -\xi_j \qquad \text{for} \qquad 1 \leq j \leq n.$$

The reflection κ_j generate a group K isomorphic to $(\mathbf{Z}_2)^n$ and $\hat{u}$ is K-invariant where the generators of K act as

$$\kappa_j(\hat{u})(\xi) = -\hat{u}(\kappa_j\xi) \qquad \text{for} \qquad 1 \leq i \leq n.$$

Now it can be shown that $\hat{u}$ is a solution of $\mathcal{P}(u) = 0$ if and only if the operator $\mathcal{P}$ is K-invariant

$$\mathcal{P}(\kappa_j(u)) = -\kappa_j(\mathcal{P}(u)) \qquad \text{for} \qquad 1 \leq i \leq n.$$

Note that the operator $\mathcal{P}$ is K-invariant if and only if F is an odd function of u.

The analogue of theorem 3 for Dirichlet boundary conditions is as follows:

Theorem 4 *Let $\mathcal{P}$ be a K-invariant reaction-diffusion operator. Then*

1. *Every smooth K-invariant solution $\hat{u}$ of $\mathcal{P}$ on $\hat{\Omega}$ satisfying periodic boundary conditions restricts to a smooth solution of the Dirichlet problem for $\mathcal{P}$ on Ω.*

2. *Let $u \in C^1(\Omega)$ be a solution to the Dirichlet problem for $\mathcal{P}$ on Ω. Then*

    - *u is smooth.*
    - *u extends uniquely to a smooth K-invariant solution of $\mathcal{P}$ on $\hat{\Omega}$ satisfying periodic boundary conditions.*

Proof Similar to that of theorem 3 except that the action of K is slightly different.
□

2.3 Single Mode Bifurcations

The symmetries of $\mathcal{P}(u) = 0$ with PBC form the group $\mathbf{O}(2)^n$. Recall that $L = d\mathcal{P}$ denotes the linearization about $u = 0$ at $\lambda = 0$. By Golubitsky *et al.* [11], a single mode bifurcation occurs if and only if ker L is an irreducible representation of $\mathbf{O}(2)^n$. In the next section we define the group action and restrict it to Fix(K) in order to obtain the symmetry constraints imposed on the NBC problem.

2.3.1 The Group Action

The aim of this section is to give an irreducible representation $\hat{V}_k$ of $\mathbf{O}(2)^n$. Then when NBC are imposed, ker L is isomorphic to the subspace of $\hat{V}_k$ that is fixed by the action of K.

Denote $\theta = (\theta_1, \ldots, \theta_n)$ a generic n-torus element. We may write an action of $\mathbf{T}^n$ as

$$\theta : z_j \mapsto e^{i\epsilon_j \cdot \theta} z_j \qquad \text{for} \qquad 1 \leq j \leq 2^{n-1}, \tag{2.6}$$

where the ϵ_j are all elements of the form $\left(\frac{k_1}{\ell_1}, \pm\frac{k_2}{\ell_2}, \ldots, \pm\frac{k_n}{\ell_n}\right)$ for some set of integers k_j. Without loss of generality we assume that these integers are nonnegative and they are called mode numbers. Now an irreducible representation of $\mathbf{T}^n$ may be written as

$$\hat{V}_k = \text{span}\left\{e^{i\epsilon_j \cdot \xi} | 1 \leq j \leq 2^{n-1}\right\}.$$

If all the mode numbers are zero then $\hat{V}_k$ is isomorphic to $\mathbb{R}$ and the action of the n-torus is trivial. Otherwise $\hat{V}_k$ is isomorphic to $\mathbf{C}^{2^{j-1}}$ where j is the number of nonzero mode numbers. Note that the action (2.6) on $\hat{V}_k$ is naturally induced by the translations of ξ_j defined before. There is also a well-defined action of K on $\hat{V}_k$ induced by the reflections κ_j of ξ_j. The generator κ_1 sends each eigenfunction to the complex conjugate of another one and the remaining κ_j act by permuting eigenfunctions.

We have just proved that the periodic extension property implies the existence of a set of n mode numbers associated to each solution of the steady reaction-diffusion equation $\mathcal{P}(u) = 0$. This is the analog of part 1 of proposition 1 in chapter 1 for more general domains. In chapter 1 we consider the domain as an interval and here we generalize to n-dimensional rectangles. The result is as follows:

Proposition 2 *Under the above hypotheses on $\mathcal{P}(u) = 0$, satisfying Neumann boundary conditions, we have that bifurcating solutions have a well-defined set of mode numbers $\underline{k} \in \mathbf{N}^n$.*

Proof By theorem 3 there is a bifurcation at $\lambda = 0$ in equilibrium solutions of $\mathcal{P}(u) = 0$ with PBC on the larger rectangle $\hat{\Omega}$. The group of symmetries of this bifurcation problem is $\mathbf{O}(2)^n$. The result comes from the definition of the $\mathbf{T}^n$-action (2.6) above. □

2.3.2 Boundary Conditions as Symmetry Constraints

As explained above the NBC problem is obtained by restricting a system with $O(2)^n$-symmetry to the fixed point subspace $\mathrm{Fix}(K)$. In this way the NBC may be seen as symmetry constraints on the periodic boundary value problem. The group action restricted to $\mathrm{Fix}(K)$ is the quotient $N_{O(2)^n}(K)/K$ where $N_{O(2)^n}(K)$ is the normalizer of K in $O(2)^n$. The result of this restriction is given in proposition 3 below. Note that this generalizes, in a very natural way, part 2 of proposition 1 in chapter 1.

Proposition 3 *Under the above hypotheses on $\mathcal{P}(u) = 0$, satisfying Neumann boundary conditions, we have that generically, if at least one of the mode numbers k_j is positive, the bifurcation is a pitchfork.*

Proof The group of symmetries of the bifurcation problem with PBC on $\dot{\Omega}$ is $O(2)^n$ and an irreducible representation of this group may be written as $\dot{V}_k$ defined before. If NBC are imposed we have that ker L is isomorphic to the subspace of $\dot{V}_k$ that is fixed by K. We denote this subspace by V_k. A direct calculation shows that

$$V_k \;=\; \mathrm{span}\left\{\cos\left(\frac{k_1\xi_1}{\ell_1}\right)\cdots\cos\left(\frac{k_n\xi_n}{\ell_n}\right)\right\}$$

which is isomorphic to $\mathbb{R}$.

Now assume that $k_j > 0$. Then the solution is $\frac{2\pi\ell_j}{k_j}$-periodic in the direction ξ_j of the domain and a translation of half period in this direction sends the only eigenfunction to its negative. Thus bifurcating solutions have a conjugate by the group $\mathbf{Z}_2$. □

2.4 Interaction of 2 Modes

Now assume that $\mathcal{P}$ depends on the additional parameter r and that $\mathcal{P}$ undergoes a simultaneous bifurcation of two steady states with mode numbers $\underline{k}, l \in \mathbf{N}^n$ when the parameters are set to zero.

2.4.1 Group Theory

Under the assumption of PBC we have that ker L is isomorphic to the direct sum

$$\dot{V}_k \oplus \dot{V}_l,$$

where $\dot{V}_k$ and $\dot{V}_l$ are two irreducible representations of $O(2)^n$ as defined in the previous section. By inspection on the action of $\mathbf{T}^n$ we see that there is no loss of generality in assuming that k_j and l_j are coprime. Otherwise we factor out the kernel of the group action.

Denote $N_n = N_{O(2)^n}(K)/K$ and H_n the quotient of N_n by the kernel of its action on $\mathrm{Fix}(K)$. By theorem 5 below the action of H_n on $\mathrm{Fix}(K)$ imposes some symmetries on the NBC problem. However these symmetries are not sufficient to guarantee an extension to a $O(2)^n$-invariant mode interaction. Sufficient conditions will be given explicitly in section 2.4.2 by invariant theory methods. We end this section by stating and proving the following:

Theorem 5 *Let $\mathbf{O}(2)^n$ act on $\hat{V}_k \oplus \hat{V}_l$ as before. Let V_k and V_l be the subspace of $\hat{V}_k$ and $\hat{V}_l$ respectively, that is fixed by K. Then $V_k \oplus V_l$ may be identified with $\mathbb{R}^2$ and H_n is isomorphic to*

- *$\mathbf{Z}_2$ if all k_j have the same parity and all l_j have the same parity;*
- *$\mathbf{Z}_2 \oplus \mathbf{Z}_2$ otherwise.*

Proof It is well known that if Σ is a subgroup of Γ then $N_\Gamma(\Sigma)$ is the largest subgroup of Γ that leaves Fix Σ setwise invariant.

Let $\mathbf{O}(2)^n$ act diagonally on $\hat{V}_k \oplus \hat{V}_l$ where the action on each block is as in section 2.3. Denote by $V_k \oplus V_l$ the subspace fixed by the reflections K. By inspection on the action of K we see that both V_k and V_l are isomorphic to $\mathbb{R}$. Under the assumption that k_j and l_j are coprime we have that N_n is generated by translations of $\pi\ell_j$ as

$$\nu_j : (x, y) \mapsto ((-1)^{k_j} x, (-1)^{l_j} y) \qquad \text{for} \qquad 1 \leq j \leq n.$$

All the ν_j act nontrivially because k_j and l_j are coprime. Thus H_n is at least $\mathbf{Z}_2$ and it is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ unless all k_j have the same parity and all l_j have the same parity (which may differ from that of the k_j). □

2.4.2 Invariant Theory

In this section we give sufficient conditions under which a mapping in $\mathbb{R}^2$ may be extended to an $\mathbf{O}(2)^n$-equivariant. The conditions in proposition 5 are necessary but not sufficient. We begin by showing that the $\mathbf{O}(2)^n$-equivariants restricted to Fix(K) may be obtained directly from the invariants. Then the form of the most general equivariant will be determined by the generators of the invariants. We need a theorem that generalizes a result of Hill and Stewart [13].

Theorem 6 *Let $\mathbf{T}^n$ act diagonally on $\mathbf{C}^m$, and $I_1, \ldots I_r$ generate the $\mathbf{C}$-valued invariants. Then the equivariants are generated by the mappings*

$$\begin{array}{cc} & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \end{pmatrix} \\ \textit{row } j \rightarrow & \begin{pmatrix} \frac{\partial I_g}{\partial \bar{z}_j} \end{pmatrix} \\ & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{array}$$

for $1 \leq j \leq m$ and $1 \leq g \leq r$.

Proof Let $(E_1, \ldots, E_m)^t$ be an equivariant, without loss of generality with monomial components (the superscript t means transposed). Then a direct computation shows that $\bar{z}_j E_j$ is invariant for all j. Therefore we may write

$$\bar{z}_j E_j = I_{g_1} I_{g_2} \cdots I_{g_k}.$$

Then $\bar{z}_j$ divides some I_g, without loss of generality I_{g_1}. Therefore

$$\begin{aligned} E_j &= (\bar{z}_j^{-1} I_{g_1}) I_{g_2} \cdots I_{g_s} \\ &= E_j^{g_1} I_{g_2} \cdots I_{g_s}. \end{aligned}$$

The result follows by observing that $\bar{z}_j^{-1} I_{g_1}$ is a rational multiple of $\frac{\partial I_{g_1}}{\partial \bar{z}_j}$. □

Armbruster and Dangelmayr [1, 6] computed the O(2)-invariants and equivariants and then restricted the result to Fix(K). Gomes [12] tried the same method and found that the generators of the $\mathbf{O}(2)^2$-invariants had to be computed algorithmically. We expect that to follow the same method for $\mathbf{O}(2)^n$ should lead to a very complicated problem. This is the reason for the different approach that will be used in this section. We compute the restricted version of the invariants directly from the information given by the defining conditions for $\mathbf{O}(2)^n$-invariance. It is well known (see Golubitsky *et al.* [11]) that given a group Γ, the Γ-invariants form a ring. What we need here is a variation of this result. We have a subgroup Σ of Γ and we want the restriction of the Γ-invariants to the fixed point subspace Fix(Σ). It still holds that the restricted invariants form a ring. Our aim is to achieve the form of a generic equivariant. Again by Golubitsky *et al.* [11] the Γ-equivariants form a module over the ring of the invariants. It is also true that the restricted equivariants form a module over the ring of the restricted invariants. This will give the form of the generic bifurcation equation under the imposed symmetry constraints. We have the following:

Theorem 7 *Let* $\mathbf{O}(2)^n$ *act diagonally on* $\dot{V}_k \oplus \dot{V}_l$ *as before and* $I_1, \ldots I_r$ *generate the invariants restricted to Fix*(K). *Then the equivariants restricted to Fix*(K) *are generated by the mappings*

$$\begin{pmatrix} \frac{\partial I_g}{\partial x} \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ \frac{\partial I_g}{\partial y} \end{pmatrix}.$$

for $1 \leq g \leq r$.

Proof This is a consequence of theorems 6 and 5. □

We proceed by finding a minimal set of generators for the invariants restricted to Fix(K). The result is as follows:

Theorem 8 *Let the group* $\mathbf{O}(2)^n$ *act diagonally on* $\dot{V}_k \oplus \dot{V}_l$ *as before. Then the* $\mathbf{O}(2)^n$*-invariants restricted to Fix*(K) *are generated according to the mode numbers as follows:*

- *If all* k_j *have the same parity and all* l_j *have the same parity the generators are* x^2, y^2 *and* $x^l y^k$, *where* $k = max_j k_j$ *and* $l = max_j l_j$.
- *Otherwise the generators are* x^2, y^2.

Before proving theorem 8 we state and prove the following:

Lemma 5 *Let M_n be the $2^{n-1} \times 2^{n-1}$-matrix constructed by induction as follows:*

$$M_1 = (1), \qquad M_n = \begin{pmatrix} M_{n-1} & M_{n-1} \\ M_{n-1} & -M_{n-1} \end{pmatrix}, \qquad n \geq 2. \tag{2.7}$$

Define the lattice

$$\mathcal{M} \;=\; \left\{ c \in \mathbb{Z}^{2^{n-1}} | M_n c \equiv 0 \pmod{2^{n-1}\mathbb{Z}^{2^{n-1}}} \right\}. \tag{2.8}$$

Given $a \in \mathbb{Z}^n$ define

$$S_a \;=\; \left\{ c \in \mathbb{Z}^{2^{n-1}} | c_1 = a_1, c_{2^{j-1}+1} = a_j, 2 \leq j \leq n \right\}. \tag{2.9}$$

Then by letting $\mathcal{L}_a = \mathcal{M} \cap S_a$ we have that

a) *$\mathcal{L}_a \neq \emptyset$ if and only if $a_1 \equiv \ldots \equiv a_n \pmod 2$;*

b) *If $a \in \mathbb{N}^n$ and $a_1 \geq \cdots \geq a_n$ then $\mathcal{L}_a \neq \emptyset \Rightarrow \frac{1}{2^{n-1}} M_n(\mathcal{L}_a) \cap \mathbb{N}^{2^{n-1}} \neq \emptyset$.*

Proof

a) Let P_n be the $2^{n-1} \times 2^{n-1}$-matrix constructed by induction as follows:

$$P_1 = (1), \qquad P_n = \begin{pmatrix} 2P_{n-1} & 0 \\ P_{n-1} & -P_{n-1} \end{pmatrix}, \qquad n \geq 2.$$

Now we have that $M_n = Q_n P_n$ where Q_n is defined by induction as follows:

$$Q_1 = (1), \qquad Q_n = \begin{pmatrix} Q_{n-1} & -Q_{n-1} \\ 0 & Q_{n-1} \end{pmatrix}, \qquad n \geq 2.$$

By noting that Q_n is unimodular we have that the lattice $\mathcal{M}$ in (2.8) can be defined equivalently by

$$\mathcal{M} \;=\; \left\{ c \in \mathbb{Z}^{2^{n-1}} | P_n c \equiv 0 \pmod{2^{n-1}\mathbb{Z}^{2^{n-1}}} \right\}. \tag{2.10}$$

We see by induction that for $0 \leq j \leq n-2$ in the 2^j+1-row of the of the matrix P_n exactly two entries are nonzero:

$$\begin{aligned} \text{entry in column } 1 &= 2^{n-2} \\ \text{entry in column } 2^j + 1 &= -2^{n-2}. \end{aligned}$$

Thus, in order for $c \in S_a$ to belong to $\mathcal{M}$ it is necessary that $a_1 \equiv \ldots \equiv a_n \pmod 2$. The sufficiency of this condition comes from the fact that P_n is a triangular matrix whose diagonal entries are nonzero.

b) From the definitions (2.8,2.10) of the lattice $\mathcal{M}$ we see that all the mappings between lattices in the following diagram are bijections:

$$\begin{tikzcd}
\mathbb{Z}^{2^{n-1}} \arrow[rr, "Q_n"] & & \mathbb{Z}^{2^{n-1}} \\
& \mathcal{M} \arrow[ul, "\frac{1}{2^{n-1}}P_n"] \arrow[ur, "\frac{1}{2^{n-1}}M_n"'] &
\end{tikzcd}$$

By induction on n we see that

$$M_n M_n^T = 2^{n-1} I, \qquad P_n' P_n = 2^{n-1} I,$$

where P_n' is defined by induction as follows:

$$P_1' = (1), \qquad P_n' = \begin{pmatrix} P_{n-1}' & 0 \\ P_{n-1}' & -2P_{n-1}' \end{pmatrix}, \qquad n \geq 2.$$

Therefore the inverse of the mappings from $\mathcal{M}$ onto $\mathbb{Z}^{2^{n-1}}$ in the diagram above are

$$\left(\frac{1}{2^{n-1}} M_n\right)^{-1} = M_n, \qquad \left(\frac{1}{2^{n-1}} P_n\right)^{-1} = P_n'.$$

Recall that $\mathcal{L}_a$ is a codimension n subset of $\mathcal{M}$ obtained by fixing n components as in (2.9). We want to show that if $\mathcal{L}_a$ is nonempty then its image under $\frac{1}{2^{n-1}} M_n$ intersects $\mathbb{N}^{2^{n-1}}$. This is the same as showing that

$$\frac{1}{2^{n-1}} Q_n P_n(\mathcal{L}_a) \cap \mathbb{N}^{2^{n-1}} \neq \emptyset.$$

or equivalently

$$\frac{1}{2^{n-1}} P_n(\mathcal{L}_a) \cap Q_n^{-1}(\mathbb{N}^{2^{n-1}}) \neq \emptyset. \tag{2.11}$$

Now we consider the hypothesis $a_1 \geq \cdots \geq a_n$ and find one particular element in the intersection (2.11). Let u be an arbitrary element of $\mathbb{Z}^{2^{n-1}}$. If $\mathcal{L}_a \neq \emptyset$ we have that $P'(u) \in \mathcal{L}_a$ if and only if

$$u_1 = a_1, \qquad u_1 - 2u_{2^{j-2}+1} = a_j, \qquad 2 \leq j \leq n$$

which is the same as

$$u_1 = a_1, \qquad u_{2^{j-2}+1} = \tfrac{a_1 - a_j}{2}, \qquad 2 \leq j \leq n. \tag{2.12}$$

The components of u not involved in (2.12) are free to take any values. We want u to satisfy also $Q_n(u) \in \mathbb{N}^{2^{n-1}}$. For that we choose the free components of u as follows:

$$u_{2^{j-2}+i} = u_{2^{j-2}+1}, \qquad 2 \leq j \leq n, \quad 2 \leq i \leq 2^{j-2}.$$

Now the element $\gamma = Q_n(u)$ belongs to $\frac{1}{2^{n-1}} M_n(\mathcal{L}_a) \cap \mathbb{N}^{2^{n-1}}$ if the hypothesis in the lemma are satisfied and the result follows. $\square$

Proof of theorem 8 We give explicit conditions for $\mathbf{O}(2)^n$-invariance and interpret them in $\mathrm{Fix}(K)$. The ring of $\mathbf{T}^n$-invariants is generated by certain monomials represented in multi-index notation by

$$z^{\underline{\alpha}^1}\bar{z}^{\underline{\beta}^1}w^{\underline{\alpha}^2}\bar{w}^{\underline{\beta}^2} \tag{2.13}$$

where $z \in \tilde{V}_k$, $w \in \tilde{V}_l$ and $\underline{\alpha}^1, \underline{\beta}^1, \underline{\alpha}^2, \underline{\beta}^2 \in \mathbf{N}^{2^{n-1}}$. Set $\underline{\gamma} = \underline{\alpha}^1 - \underline{\beta}^1, \underline{\delta} = \underline{\alpha}^2 - \underline{\beta}^2 \in \mathbf{Z}^{2^{n-1}}$ and the $n \times 2^{n-1}$-matrix L_n, where the entries in column j are the signs of the components of ϵ_j in the same order and the ϵ_j are all elements of the form $\left(\frac{k_1}{l_1}, \pm\frac{k_2}{l_2}, \ldots, \pm\frac{k_n}{l_n}\right)$ induced by the $\mathbf{T}^n$-action (2.6) in section 2.3. Then the condition for $\mathbf{T}^n$-invariance becomes

$$\begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix} L_n\underline{\gamma} + \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_n \end{pmatrix} L_n\underline{\delta} \;=\; 0. \tag{2.14}$$

We set

$$\underline{a} = L_n\underline{\gamma} \qquad \underline{b} = L_n\underline{\delta}.$$

and (2.14) can be represented by

$$a_jk_j + b_jl_j = 0, \qquad 1 \leq j \leq n.$$

Now let M_n be the $2^{n-1} \times 2^{n-1}$-matrix defined in (2.7) and apply the coordinate change

$$\underline{c} = M_n\underline{\gamma}, \qquad \underline{d} = M_n\underline{\delta}. \tag{2.15}$$

By induction on n we see that

$$M_n = M_n^t, \qquad M_n^2 = 2^{n-1}I.$$

Therefore

$$\underline{\gamma} = \frac{1}{2^{n-1}}M_n\underline{c}, \qquad \underline{\delta} = \frac{1}{2^{n-1}}M_n\underline{d}. \tag{2.16}$$

By reordering the columns of L_n if necessary we have that $\underline{a}, \underline{b}$ are projections of $\underline{c}, \underline{d}$ onto $\mathbf{Z}^n$ such that

$$\begin{array}{llll} a_1 = c_1 & & b_1 = d_1 & \\ a_j = c_{2^{j-2}+1}, & 2 \leq j \leq n & b_j = d_{2^{j-2}+1}, & 2 \leq j \leq n. \end{array} \tag{2.17}$$

We want to find necessary and sufficient conditions under which a given $\underline{a} \in \mathbf{Z}^n$ may be lifted as $\underline{c} \in \mathbf{Z}^{2^{n-1}}$ such that $\underline{\gamma} \in \mathbf{Z}^{2^{n-1}}$ where $\underline{\gamma}$ is as in (2.16). This is achieved if and only if

$$M_n\underline{c} \equiv 0 \pmod{2^{n-1}\mathbf{Z}^{2^{n-1}}}.$$

The same condition will then be imposed on $\underline{b}$. By lemma 5.a) the answer is

$$a_1 \equiv \ldots \equiv a_n \pmod 2.$$

The trivial solution of (2.15) corresponds to elements generated by

$$z_j\bar{z}_j, \qquad w_j\bar{w}_j, \qquad 1 \leq j \leq 2^{n-1}. \tag{2.18}$$

Dividing out factors of these from (2.13) leads to a monomial in multi-index notation as

$$u^{|\underline{\gamma}|}v^{|\underline{\delta}|} \tag{2.19}$$

where $|\,.\,|$ denotes the modulus of each component and u_j is z_j or $\bar{z}_j$, v_j is w_j or $\bar{w}_j$ depending on the sign of the j-component of $\underline{\gamma}$, $\underline{\delta}$ respectively.

In theorem 5 we saw that Fix(K) is isomorphic to $\mathbb{R}^2$. Thus the invariants (2.18,2.19) become

$$N_1 = x^2, \qquad N_2 = y^2, \qquad T = x^\gamma y^\delta,$$

where γ and δ represent the sum of the moduli of each component of $\underline{\gamma}$ and $\underline{\delta}$ respectively. Now solutions of (2.15) are of the form $(\underline{a},\underline{b}) = (A\underline{l}, -A\underline{k})$ where A is a diagonal matrix with integer entries. Under the assumption that hcf $(k_j,l_j) = 1$ for $1 \leq j \leq n$ above we have that all the diagonal entries of A must have the same parity. By imposing condition (2.18) to $\underline{a}, \underline{b}$, we get the diagonal entries of A may be odd only if

$$k_1 \equiv \ldots \equiv k_n \pmod 2, \qquad l_1 \equiv \ldots \equiv l_n \pmod 2. \tag{2.20}$$

From (2.15,2.17) we see that γ has the same parity as all the a_j and δ has the same parity as all the b_j. Thus γ, δ must be both even if (2.20) is not satisfied, in which case N_1, N_2 generate the ring of invariant functions. So from now on we assume that (2.20) is true. In this case there are elements of the form T that are not generated by N_1, N_2 and for these γ and all the l_j have the same parity and δ and all the k_j have the same parity.

Also from (2.15,2.17) we have that

$$\gamma \geq \max_j |a_j|, \qquad \delta \geq \max_j |b_j|. \tag{2.21}$$

We claim that there exist $\underline{c}, \underline{d} \in \mathbb{Z}^{2^{n-1}}$ satisfying the equalities in (2.21). This will imply that

$$\min \gamma = \max_j |a_j|, \qquad \min \delta = \max_j |b_j|. \tag{2.22}$$

To prove the claim we assume without loss of generality that $\underline{a}, -\underline{b} \in \mathbb{N}^n$. By induction on n we have that all the entries in the first column of M_n are $+1$ and in the other columns the number of $+1$s and -1s is the same. For simplicity we assume that $a_1 \geq \cdots \geq a_n$. Thus in order to get the first equality in (2.22) it is enough to show that $\underline{a}$ can be lifted as $\underline{c} \in \mathbb{Z}^{2^{n-1}}$ such that $\underline{\gamma} \in \mathbb{N}^{2^{n-1}}$, where $\underline{\gamma} = M_n^{-1}\underline{c}$.

This comes immediately from lemma 5b). Now let π be a permutation such that $-b_{\pi^{-1}(1)} \geq \cdots \geq -b_{\pi^{-1}(n)}$ and apply a variation of lemma 5 where M_n is substituted by $\pi^{-1} M_n \pi$ to show that the two equalities (2.22) hold simultaneously.

Now we are left with minimizing the absolute values of the components of $\underline{a}, \underline{b}$ when the diagonal entries of A are all odd. Recalling that

$$(\underline{a}, \underline{b}) = (A\underline{l}, -A\underline{k})$$

we see that all the components achieve their minimum values simultaneously when A is the identity and the result follows. □

Now we have everything we need in order to give the main final result, which gives the form of the bifurcation equations for steady-state mode interactions in reaction-diffusion equations on generalized rectangles with NBC.

Theorem 9 *Assume that $\mathcal{P}(u) = 0$ undergoes a simultaneous bifurcation of the two modes $\underline{k}, \underline{l} \in \mathbb{N}$ from $u = 0$ when the parameter λ crosses zero. Then the form of the reduced bifurcation equations on $\ker L$ depends on the mode numbers as follows:*

1. *If all k_j have the same parity and all l_j have the same parity we have*

$$\begin{aligned} ax + cx^{l-1}y^k &= 0 \\ by + dx^l y^{k-1} &= 0 \end{aligned}$$

   where $k = max_j k_j$ and $l = max_j l_j$.

2. *Otherwise*

$$\begin{aligned} ax &= 0 \\ by &= 0 \end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ.

Proof This comes directly from theorems 7 and 8. □

In case 1 the bifurcation equations reduce to those of Armbruster and Dangelmayr [1] when $n = 1$. However, their assumption that k and l are coprime is no longer always valid. In case 2 the equations are $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant and those have been studied by Golubitsky *et al.* [11]. In this case there are no degeneracies created by the periodic extension.

2.5 Interaction of m Modes

Now assume that $\mathcal{P}$ depends on $m - 1$ additional parameters. Then a simultaneous bifurcation for the parameter λ of m steady states with distinct mode numbers is generic. Suppose that $\mathcal{P}$ undergoes a simultaneous bifurcation of m steady states with mode numbers $\underline{k}^1, \ldots, \underline{k}^m \in \mathbb{N}^n$ when the parameters are set to zero. The methods introduced in section 2.4 on interaction of two modes apply almost directly to the case where any number of modes bifurcate simultaneously. However, the final result here is not as complete as in the previous section in the sense that a minimal set of generators of the restricted equivariants will not be given. The invariant theory involves more complicated combinatorics if we have more than two modes interacting. In this section we give a nonminimal set of generators of the invariants. After this, a minimal set can be computed algorithmically. As before the equivariants may be obtained directly from the invariants.

2.5.1 Group Theory

Under the assumption of PBC we have that ker L is isomorphic to $\oplus_{j=1}^{m} V_{k^j}$ where each block is an irreducible representation of $\mathbf{O}(2)^n$ as in section 2.3. We may factor out the kernel of the group action and assume that

$$\mathrm{hcf}(k_j^1, \dots, k_j^m) = 1 \qquad \text{for} \qquad 1 \leq j \leq n. \tag{2.23}$$

As in section 2.4.1 we denote $N_n = \mathrm{N}_{\mathbf{O}(2)^n}(K)/K$ and H_n the quotient of N_n by the kernel of its action on $\mathrm{Fix}(K)$. Theorem 10 gives the symmetries imposed by the action of H_n on $\mathrm{Fix}(K)$ on the NBC problem. Before stating and proving this result we need some preliminary notation. Given a set of mode numbers k we denote by p_k a vector with components

$$p_k[j] = (-1)^{k_j}.$$

Theorem 10 *Let* $\mathbf{O}(2)^n$ *act diagonally on* $\oplus_{j=1}^{m} V_{k^j}$ *as before. Then* $\mathit{Fix}(K)$ *may be identified with* $\mathbf{R}^m$ *and* H_n *is isomorphic to* $(\mathbf{Z}_2)^q$ *where* q *in the rank of the matrix* $(p_{k^1}, \dots, p_{k^m})$.

Proof This proof is very similar to that of theorem 5 for the interaction of 2 modes. We find that N_n is generated by

$$\kappa_j: \; x \mapsto ((-1)^{k_j^1} x_1, \dots, (-1)^{k_j^m} x_m), \qquad 1 \leq j \leq n. \tag{2.24}$$

This is a group isomorphic to $(\mathbf{Z}_2)^q$ where q is the number of independent reflections. More precisely q is the rank of the matrix with the vectors p_{k^j} for columns. □

2.5.2 Invariant Theory

In this section we give a nonminimal set of generators of the invariants under the $\mathbf{T}^n$-action on $\oplus_{j=1}^{m} V_{k^j}$ defined above. We leave the problem in a suitable state for the application of an algorithm giving a minimal set of generators.

By restricting the $\mathbf{T}^n$-equivariants given by theorem 6 to $\mathrm{Fix}(K)$ where K acts on each block as in section 2.3 we get

Theorem 11 *Let $O(2)^n$ act on $\oplus_{j=1}^{m}\hat{V}_{k_j}$ as before and $I_1,\dots I_r$ generate the invariants restricted to $Fix(K)$. Then the equivariants restricted to $Fix(K)$ are generated by the mappings*

$$\begin{array}{cc} & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial I_g}{\partial x_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \text{row } j \rightarrow & \end{array}$$

for $1 \le j \le m$ and $1 \le g \le r$.

Proof This is a consequence of theorems 6 and 10. □

Theorem 12 *Let the group $O(2)^n$ act on $\oplus_{j=1}^{m}\hat{V}_{k_j}$ as before. Then the $\mathbf{T}^n$-invariants restricted to $Fix(K)$ are generated by $x_1^2,\dots,x_m^2$ together with monomials of the form*

$$x_1^{a^1}\cdots x_m^{a^m}$$

where $a^i = max_j|a^i_j|$ and the a^i_j satisfy the conditions

- $a^1_j k^1_j + \cdots + a^m_j k^m_j = 0, \qquad 1 \le j \le n$
- $a^i_1 \equiv \dots \equiv a^i_n \pmod 2 \qquad$ *for* $1 \le i \le m$

Proof The ring of $\mathbf{T}^n$-invariants is generated by monomials represented in multi-index notation by

$$Z_1^{\underline{\alpha}^1}\overline{Z}_1^{\underline{\beta}^1}\cdots Z_m^{\underline{\alpha}^m}\overline{Z}_m^{\underline{\beta}^m} \tag{2.25}$$

where $Z_i \in \hat{V}_{k^i}$ and $\underline{\alpha}^i, \underline{\beta}^i \in \mathbf{N}^{2^{n-1}}$. Setting $\underline{\gamma}^i = (\underline{\alpha}^i - \underline{\beta}^i) \in \mathbf{Z}^{2^{n-1}}$ for $1 \le i \le m$ we have that the condition for $\mathbf{T}^n$-invariance is

$$a^1_j k^1_j + \cdots + a^m_j k^m_j \;=\; 0, \qquad 1 \le j \le n \tag{2.26}$$

where $\underline{a}^i = (a^i_1,\dots,a^i_n) \in \mathbf{Z}^n$ is such that

$$\underline{a}^i = L\underline{\gamma}^i, \qquad 1 \le i \le r \tag{2.27}$$

and L_n is the $n \times 2^{n-1}$-matrix introduced in section 2.4.2 on interaction of two modes. We want to find the conditions on $\underline{a}^i \in \mathbf{Z}^n$ for which there exists a $\underline{\gamma}^i \in \mathbf{Z}^{2^{n-1}}$ satisfying (2.27).

In order to use lemma 5 of section 2.4.2 recall that M_n is the $2^{n-1} \times 2^{n-1}$-matrix defined in (2.7) as

$$M_1 = (1), \qquad M_n = \begin{pmatrix} M_{n-1} & M_{n-1} \\ M_{n-1} & -M_{n-1} \end{pmatrix}, \qquad n \ge 2$$

and apply the coordinate change

$$\underline{c}^i = M_n \gamma^i \qquad \text{for } 1 \le i \le r. \tag{2.28}$$

By reordering the columns of L_n if necessary we have that $\underline{a}^i$ is a projection of $\underline{c}^i$ onto $\mathbb{Z}^n$ such that

$$a_1^i = c_1^i \qquad \text{and} \qquad a_j^i = c_{2^{j-2}+1}^i \qquad \text{for } 2 \le j \le n. \tag{2.29}$$

We want to find necessary and sufficient conditions under which a given $\underline{a}^i \in \mathbb{Z}^n$ may be lifted as $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ such that $\underline{\gamma}^i \in \mathbb{Z}^{2^{n-1}}$ where by (2.28)

$$\underline{\gamma}^i = \tfrac{1}{2^{n-1}} M_n \underline{c}^i.$$

This is achieved if and only if

$$M_n \underline{c}^i \equiv 0 \pmod{2^{n-1}\mathbb{Z}^{2^{n-1}}} \qquad \text{for } 1 \le i \le m$$

By lemma 5.a) the answer is

$$a_1^i \equiv \ldots \equiv a_n^i \pmod 2 \qquad \text{for } 1 \le i \le m \tag{2.30}$$

Monomials of the form $z\bar{z}$ are invariant. Dividing out factors of these from (2.25) leads to a monomial in multi-index notation as

$$U_1^{|\underline{\gamma}^1|} \cdots U_m^{|\underline{\gamma}^m|}$$

where $|\,.\,|$ denotes the modulus of each component and the jth component of $U_i \in \hat{V}_{k^i}$ is the jth component of Z_i or $\overline{Z}_i$ depending on the sign of the γ_j^i.

By restricting to $\mathrm{Fix}(K) = \mathbb{R}^m$ we get that the invariants are generated by

$$x_1^2, \ldots, x_m^2 \qquad \text{and} \qquad \prod_i x_i^{\gamma^i}$$

where γ^i is the sum of the moduli of each component of $\underline{\gamma}^i$. From (2.28,2.29) we get

$$\gamma^i \ge \max_j |a_j^i|, \qquad 1 \le i \le m. \tag{2.31}$$

We claim that there exists $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ satisfying the equalities in (2.31). This will imply that

$$\min \gamma^i = \max_j |a_j^i|, \qquad 1 \le i \le m \tag{2.32}$$

To prove the claim we assume without loss of generality that $\underline{a}^i \in \mathbb{N}^n$. By induction on n we have that all the entries in the first column of M_n are $+1$ and in the other columns the number of $+1$s and -1s is the same. For simplicity we assume that $a_1^i \ge \cdots \ge a_n^i$. Thus in order to get the first equality in (2.32) it is enough to show that $\underline{a}^i$ can be lifted as $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ such that $\underline{\gamma}^i \in \mathbb{N}^{2^{n-1}}$, where $\underline{\gamma}^i = M_n^{-1}\underline{c}^i$. This comes immediately from lemma 5b) in section 2.4. $\square$

Recall that in section 2.4, the general form for the bifurcation equations was given for interaction of two modes (see theorem 8). Unfortunately an analogue of that result cannot be obtained when an arbitrary number m of modes bifurcate simultaneously. The reason is that a minimal set of generators for the invariants (as in theorem 12) must be computed algorithmically. We wrote a computer program that works for m up to 4, which is enough for the application that we are interested in (Bénard convection, see chapter 4). The algorithm used is not described in this thesis for two reasons: first it is too complicated and second our generators can be obtained by adapting an algorithm of Fekken [7].

We proceed by stating the problem that this author deals with and describing briefly how to adapt his algorithm to our situation. Fekken wants the invariants under an action of $\mathbf{S}^1$ on $\mathbf{C}^m$ as

$$\theta : z_j \mapsto e^{ik^j\theta} z_j \qquad \text{for} \qquad 1 \leq j \leq m,$$

where the k^j are nonnegative integers and $z_j \in \mathbf{C}$. The invariants under this action of the group $\mathbf{S}^1$ are generated by $z_j\bar{z}_j$ for $1 \leq j \leq m$ together with monomials of the form

$$w_1^{|a^1|} \cdots w_m^{|a^m|},$$

where the a^j satisfy the equation

$$k^1a^1 + \cdots + k^m a^m = 0 \qquad (2.33)$$

and w_j is z_j or $\bar{z}_j$ if a^j is, respectively, ≥ 0 or < 0. Now equation (2.33) defines a codimension one lattice $\mathcal{L}$ on $\mathbf{Z}^m$. Given that w_j depends on the sign of a^j, Fekken divides the lattice $\mathcal{L}$ into cones according to the sign of the a^j and computes the generators of each cone.

In our problem instead of $w_j \in \mathbf{C}$ we have $x_j \in \mathbf{R}$, and therefore the signs af the a^j are not important. We have n equations of the form (2.33) defining n lattices $\mathcal{L}_1, \ldots, \mathcal{L}_n$. We apply the algorithm of Fekken to each equation independently and then take the modulus of the result. We may get some redundancies that can be eliminated. Then the remaining elements must be distributed into classes according to their parities. Inside each class we compute the maxima as in theorem 12.

Chapter 3

Permutations of Edges as Additional Symmetries

3.1 Introduction

As in chapter 2, also here we are concerned with reaction-diffusion equations on the euclidean space $\mathbb{R}^n$. The difference is that now the domain is an n-dimensional cube. all the edges have the same length. This introduces extra symmetries into the problem studied previously.

Let $\mathcal{P}(u)$ denote a reaction-diffusion equation on the cube $[0,\pi]^n$ with NBC. As before all the steady solutions of this problem extend to the bigger cube $[-\pi,\pi]^n$ by reflection across the boundaries, giving a solution of the same equations but satisfying PBC on the larger domain. Note that, by construction, these new solutions are invariant under reflection across the boundaries of $[0,\pi]^n$. Conversely, all solutions of the PBC problem that are invariant under these reflections restrict to a solution of the NBC problem. Therefore, the original problem has all the symmetries of the PBC problem that keep the NBC on $[0,\pi]^n$, but not necessarily leave this domain invariant.

Up to now it all works just as it did for the rectangle (as expected because the cube is a special rectangle). A cube is distinguished from an ordinary rectangle by being invariant under permutations of edges. These symmetries leave $\mathcal{P}(u)$ invariant, and also both periodic and Neumann boundary conditions. Thus, the group of symmetries of the PBC problem is now $\mathbf{O}(2)^n \dot{+} \mathbf{S}_n$, where $\mathbf{S}_n$ is the group of permutations of n elements.

In this chapter we are interested in the simplest steady-state bifurcations with the symmetries described above. We assume that $\mathcal{P}(u) = 0$ has a trivial solution $u = 0$, which undergoes a steady-state bifurcation at $\lambda = 0$. By performing a Liapunov-Schmidt reduction, $\mathcal{P}(u)$ is projected onto ker $d\mathcal{P}$ and the qualitative behaviour is maintained. Now the dimension of ker $d\mathcal{P}$ with NBC is $\leq n!$, which is the number of permutations of n elements. The simplicity of the reduced bifurcation equations obtained for the generic rectangle is very far from being observed here. This problem is really much more complicated and depends on n. General results are very hard to obtain.

This chapter is divided into five sections. In section 3.2 we define the action of

$O(2)^n \dot{+} S_n$ on ker $d\mathcal{P}$ and its restriction to the subspace where the NBC problem lives. In section 3.3, by invariant theory methods, we describe the form of the bifurcation equations projected onto ker $d\mathcal{P}$. In sections 3.4, 3.5, 3.6 we work with more detail on cubes of dimension 1,2,3 respectively. It will be seen that the problem is already complicated for $n = 3$.

As we said before, this chapter is concerned only with the simplest type of steady-state bifurcations, when all bifurcating solutions are conjugate under an action of $O(2)^n \dot{+} S_n$. Recall that more complicated bifurcations, mode interactions, have been studied for the interval, $n = 1$ (see chapter 1). In chapter 4 analogous results in the case $n = 2$, square, are needed for the 3-dimensional Bénard convection.

3.2 The Group Action

As in previous chapters, $\mathcal{P}(u)$ denotes a reaction-diffusion equation on $\mathbb{R}^n$

$$\mathcal{P} \equiv \Delta u + F(u, \lambda) \quad = \quad 0$$

and we are interested in the solutions $u(\xi)$ that satisfy NBC

$$\frac{\partial u}{\partial \xi_i}(\xi) = 0 \qquad \text{when} \qquad \xi_i = 0, \pi \qquad \text{for} \qquad 1 \leq i \leq n$$

on the generalized cube $[0, \pi]^n$. The extension method described in section 2.2 applies directly to this case because the cube is just a special rectangle. The only difference is that the domain has now a richer group of symmetries and they are preserved by the periodic extension.

Now the group of symmetries of the PBC problem on $[-\pi, \pi]^n$ is $O(2)^n \dot{+} S_n$ in contrast with a generic rectangle, which has only $O(2)^n$-symmetry. Again we assume that $\mathcal{P}(u) = 0$ has a trivial solution $u = 0$ which undergoes a steady-state bifurcation at $\lambda = 0$. Recall that $L = d\mathcal{P}$ denotes the linearization about $u = 0$ at $\lambda = 0$ and L is generically an irreducible representation of $O(2)^n \dot{+} S_n$. In order to make the group action explicit we begin by recalling the representation of $\mathbf{T}^n$ introduced in section 2.3.1

$$\theta : z_j \mapsto e^{i\kappa_j \cdot \theta} z_j$$

where the κ_j are elements of the form $(k_1, \pm k_2, \dots, \pm k_n)$ for some set of nonnegative integers k_j. These integers are called mode numbers. We write an irreducible representation of $\mathbf{T}^n$ as

$$\hat{V}_k \quad = \quad \text{span}\left\{ e^{i\kappa_j \cdot \xi} \middle| 1 \leq j \leq 2^{n-1} \right\}.$$

Let the action of K be induced by the reflections κ_j of ξ_j. Now denote s_j for $1 \leq j \leq n!$ the elements of the group of permutations S_n and construct a representation of $O(2)^n \dot{+} S_n$ as the direct sum

$$\oplus_{j=1}^{n!} \hat{V}_{s_j(k)}.$$

We define an action of $\mathbf{S}_n$ as being induced by the action of this group on the mode numbers. Note that the action of $\mathbf{O}(2)^n$ makes each block irreducible and the group $\mathbf{S}_n$ permutes them in such a way that the direct sum is irreducible if and only if all mode numbers are different, which will be assumed throughout this section.

By factoring out the kernel of the group action if necessary we may assume without loss of generality that the k_j are coprime. By restricting the group action to $\mathrm{Fix}(K)$ we obtain a result that has the line as a particular case. Denote by N_n the quotient group $N_{\mathbf{O}(2)^n\dot{+}\mathbf{S}_n}(K)/K$ and H_n the quocient of N_n by the kernel of its action on $\mathrm{Fix}(K)$.

Theorem 13 *Let $\mathbf{O}(2)^n\dot{+}\mathbf{S}_n$ act on $\oplus_{j=1}^{n!}\hat{V}_{s_j(k)}$ as before. Then $Fix(K)$ may be identified with $\mathbb{R}^{n!}$ and H_n is isomorphic to $(\mathbf{Z}_2)^p\dot{+}\mathbf{S}_n$ if $p-1$ is the number of even mode numbers.*

Proof Inspection of the action of K, as in chapter 2, gives the identification of $\mathrm{Fix}(K)$ with $\mathbb{R}^{n!}$. Given that the k_j are assumed to be coprime, the group N_n is generated by translations of π as

$$\nu_j : \quad x \quad \mapsto ((-1)^{s_1(k_j)}x_1, \ldots, (-1)^{s_{n!}(k_j)}x_n), \qquad 1 \leq j \leq n \tag{3.1}$$

together with the restriction of $\mathbf{S}_n$ to $\mathrm{Fix}(K)$.

In order to find a set of generators for the group H_n we assume that the elements of $\mathbf{S}_n$ have been ordered so that

$$\begin{aligned} s_j &= (1 \quad j), \qquad 1 \leq j \leq n \\ s_j &\in \langle s_2, \ldots, s_n \rangle, \qquad n+1 \leq j \leq n! \end{aligned}$$

It can be seen that the induced permutations of blocks are such that

$$\begin{aligned} s_1 &= I \\ \nu_j &= s_j^{-1}\nu_1 s_j \\ &= s_j\nu_1 s_j, \qquad 2 \leq j \leq n \\ s_j &\in \langle s_2, \ldots, s_n \rangle, \qquad n+1 \leq j \leq n! \end{aligned}$$

Thus

$$\begin{aligned} N_n &= \langle \nu_1, s_2, \ldots, s_n \rangle \\ &= (\mathbf{Z}_2)^n\dot{+}\mathbf{S}_n. \end{aligned}$$

Now H_n is not always isomorphic to the full group of symmetry of the n-cube, $(\mathbf{Z}_2)^n\dot{+}\mathbf{S}_n$ because some of the reflections ν_j may act in the same way depending on the parities of the mode numbers. By inspection on the actions (3.1) it follows that H_n is actually isomorphic to $(\mathbf{Z}_2)^p\dot{+}\mathbf{S}_n$ if $p-1$ is the number of even mode numbers. $\square$

3.3 Invariant Theory

In this section we give the necessary and sufficient conditions under which a mapping in $\mathbb{R}^n$ may be extended to an $O(2)^n \dot{+} S_n$-equivariant. Recall that in chapter 2, for single mode bifurcations and 2-mode interactions on generic rectangles, we gave explicitly a minimal set of generators of the $O(2)^n$-invariants restricted to Fix(K). From those we could read the equivariants directly. On the contrary, here we give all the conditions for invariance and equivariance but a minimal set of generators in the general case will not be given. The reason is that this problem involves very complex combinatorics. However, if the mode numbers are given we can compute the generators of the $\mathbf{T}^n$-invariants algorithmically. We proceed by giving the conditions for $\mathbf{T}^n$-invariance. The result is as follows:

Proposition 4 *Let the group* $O(2)^n \dot{+} S_n$ *act on* $\oplus_{j=1}^{n!} \hat{V}_{s_j(k)}$ *as before. Then the* $\mathbf{T}^n$-*invariants restricted to* $Fix(K)$ *are monomials of the form*

$$x_1^{a^1} \cdots x_{n!}^{a^{n!}}$$

where $a^i = max_j |a^i_j|$ *and the* a^i_j *satisfy the conditions*

- $a_j^1 s_1(k_j) + \cdots + a_j^{n!} s_{n!}(k_j) = 0, \qquad 1 \le j \le n$
- $a^i_1 \equiv \ldots \equiv a^i_n \pmod 2 \qquad$ *for* $1 \le i \le n!$

Proof The ring of $\mathbf{T}^n$-invariants is generated by monomials represented in multi-index notation by

$$Z_1^{\underline{\alpha}^1} \bar{Z}_1^{\underline{\beta}^1} \cdots Z_{n!}^{\underline{\alpha}^{n!}} \bar{Z}_{n!}^{\underline{\beta}^{n!}} \tag{3.2}$$

where $Z_i \in V_{s_i(k)}$ and $\underline{\alpha}^i, \underline{\beta}^i \in \mathbb{N}^{2^{n-1}}$. Setting $\underline{\gamma}^i = (\underline{\alpha}^i - \underline{\beta}^i) \in \mathbb{Z}^{2^{n-1}}$ for $1 \le i \le n!$ we have that the condition for $\mathbf{T}^n$-invariance is

$$a_j^1 s_1(k_j) + \cdots + a_j^{n!} s_{n!}(k_j) \;=\; 0, \qquad 1 \le j \le n$$

where $\underline{a}^i = (a^i_1, \ldots, a^i_n) \in \mathbb{Z}^n$ is such that

$$\underline{a}^i = L\underline{\gamma}^i, \qquad 1 \le i \le n! \tag{3.3}$$

and L_n is the $n \times 2^{n-1}$-matrix introduced in section 2.4.2 on generalized rectangles. We want to find the conditions on $\underline{a}^i \in \mathbb{Z}^n$ for which there exists a $\underline{\gamma}^i \in \mathbb{Z}^{2^{n-1}}$ satisfying (3.3).

In order to use lemma 5 of section 2.4.2 recall that M_n is the $2^{n-1} \times 2^{n-1}$-matrix defined in (2.7) as

$$M_1 = (1), \qquad M_n = \begin{pmatrix} M_{n-1} & M_{n-1} \\ M_{n-1} & -M_{n-1} \end{pmatrix}, \qquad n \ge 2$$

and apply the coordinate change

$$\underline{c}^i = M_n \underline{\gamma}^i \quad \text{for } 1 \le i \le n! \tag{3.4}$$

By reordering the columns of L_n if necessary we have that $\underline{a}^i$ is a projection of $\underline{c}^i$ onto $\mathbb{Z}^n$ such that

$$a_1^i = c_1^i \qquad \text{and} \qquad a_j^i = c_{2^{j-2}+1}^i \qquad \text{for } 2 \leq j \leq n. \tag{3.5}$$

We want to find necessary and sufficient conditions under which a given $\underline{a}^i \in \mathbb{Z}^n$ may be lifted as $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ such that $\underline{\gamma}^i \in \mathbb{Z}^{2^{n-1}}$ where by (3.4)

$$\underline{\gamma}^i = \tfrac{1}{2^{n-1}} M_n \underline{c}^i.$$

This is achieved if and only if

$$M_n \underline{c}^i \equiv 0 \pmod{2^{n-1}\mathbb{Z}^{2^{n-1}}} \qquad \text{for } 1 \leq i \leq n!$$

By lemma 5.a) the answer is

$$a_1^i \equiv \ldots \equiv a_n^i \pmod 2 \qquad \text{for } 1 \leq i \leq n!$$

Monomials of the form $z\bar{z}$ are invariant. Dividing out factors of these from (3.2) leads to a monomial in multi-index notation as

$$U_1^{|\underline{\gamma}^1|} \cdots U_{n!}^{|\underline{\gamma}^{n!}|} \tag{3.6}$$

where $|\ .\ |$ denotes the modulus of each component and the jth component of $U_i \in V_{s,(\underline{k})}$ is the jth component of Z_i or $\overline{Z}_i$ depending on the sign of the γ_j^i.

By restricting to $\mathrm{Fix}(K) = \mathbb{R}^{n!}$ we get that the invariants are generated by

$$x_1^2, \ldots, x_{n!}^2 \qquad \text{and} \qquad \prod_i x_i^{\gamma^i}$$

where γ^i is the sum of the moduli of each component of $\underline{\gamma}^i$. From (3.4,3.5) we get

$$\gamma^i \geq \max_j |a_j^i|, \qquad 1 \leq i \leq n! \tag{3.7}$$

We claim that there exists $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ satisfying the equalities in (3.7). This will imply that

$$\min \gamma^i = \max_j |a_j^i|, \qquad 1 \leq i \leq n! \tag{3.8}$$

To prove the claim we assume without loss of generality that $\underline{a}^i \in \mathbb{N}^n$. By induction on n we have that all the entries in the first column of M_n are $+1$ and in the other columns the number of $+1$s and -1s is the same. For simplicity we assume that $a_1^i \geq \cdots \geq a_n^i$. Thus in order to get the first equality in (3.8) it is enough to show that $\underline{a}^i$ can be lifted as $\underline{c}^i \in \mathbb{Z}^{2^{n-1}}$ such that $\underline{\gamma}^i \in \mathbb{N}^{2^{n-1}}$, where $\underline{\gamma}^i = M_n^{-1}\underline{c}^i$. This comes immediately from lemma 5b) in chapter 2. □

A minimal set of generators for the restricted $\mathbf{T}^n$-invariants can be computed algorithmically as explained in section 2.5. In that section we also say how to generate the restricted $\mathbf{T}^n$-equivariants directly from the generators of the restricted $\mathbf{T}^n$-invariants (see theorem 11). Symmetrization of these over $\mathbf{S}_n$ gives the restricted $\mathbf{O}(2)^n \dot{+} \mathbf{S}_n$ invariants. Such symmetrization may be a combinatorially complex problem if n is large. In the following three sections we give a more detailed description of this procedure for $n = 1, 2, 3$.

3.4 1-Dimensional Domain

The only element of the group of permutations $\mathbf{S}_1$ is the identity. Thus for $n = 1$ cube and rectangle are the same. This case is very straightforward but we reproduce some of the results here for comparison with other values of n. The normalizer quotient H_1 is isomorphic to the group $\mathbf{Z}_2$ whose action is generated by

$$\nu_1 : x \mapsto -x.$$

The only isotropy subgroups are the identity and $\mathbf{Z}_2$ itself. The isotropy lattice is just

$$\begin{array}{c} \mathbf{Z}_2 \\ \uparrow \\ \mathbf{1} \end{array}$$

Recall that the extension property described before implies the existence of a well-defined mode number $k \in \mathbb{N}$. The group H_1 acts as the symmetries of the interval $\left[0, \frac{\pi}{k}\right]$. Since the bifurcating solution has odd mode number, 1, in the smaller domain, it has always a conjugate by reflection about $\frac{\pi}{2k}$ (which is the same as translation of $\frac{\pi}{k}$ in the corresponding PBC problem). Thus the generic bifurcation is a pitchfork and if k is even there is a reflection symmetry that would not be expected without the periodic extension.

The invariants are generated by x^2 and the reduced bifurcation equations are of the form

$$f(x, \lambda) = a(I, \lambda)x = 0 \tag{3.9}$$

where $I = x^2$ generates the invariants. The generic bifurcation from $x = 0$ is a pitchfork. The equation for the branch and its stability for the invariant system

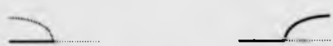
$$\dot{x} + f(x, \lambda) = 0 \tag{3.10}$$

are as follows:

Isotropy subgroup	Branching equation	Eigenvalue
$\mathbf{1}$	$a_I(0)x_1^2 + a_\lambda(0)\lambda = 0$	$a_I(0)$

Assuming that $a_\lambda(0) < 0$ the bifurcation diagram depends on the coefficient $a_I(0)$ as follows:

$a_I\,(0)<0$ $\qquad$ $a_I\,(0)>0$



3.5 2-Dimensional Domain

Crawford [3] gave a complete description of this case. Some of the results are reproduced here for uniformity of notation and comparison with other values of n. There are two cases to consider.

3.5.1 $k_1 = k_2$

In this case ker L is 1-dimensional and the action of the normalizer quotient H_2 is generated as in the previous section. The reflections ν_1, ν_2 act in the same way (non-trivially) and both permutations s_1, s_2 act trivially. Thus, as in the case $n = 1$, the generic bifurcation from $x = 0$ is a pitchfork.

3.5.2 $k_1 \neq k_2$: Group Theory

In this case the symmetric group $\mathbf{S}_2$ does not act trivially. It contains the 2 elements

$$s_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and s_2 is the unique generator. The action of H_2 is generated by

$$\begin{aligned} \nu_1 &: (x_1, x_2) \mapsto \left((-1)^{k_1} x_1, (-1)^{k_2} x_2\right) \\ s_2 &: (x_1, x_2) \mapsto (x_2, x_1) \end{aligned}$$

By theorem 13 we have that H_2 is isomorphic to

(a) $\mathbf{D}_4$ if one of k_1, k_2 is even;

(b) $\mathbf{Z}_2 \dot{+} \mathbf{S}_2$ if k_1, k_2 are both odd.

These are all the possibilities because by factoring out the kernel of the action of $\mathbf{O}(2)^2 \dot{+} \mathbf{S}_2$ on $\mathbf{C}^4$, in section 3.2 we have assumed without loss of generality that k_1, k_2 are coprime. From now on we denote

$$\mathbf{Z}_k^1 = \left\langle (\theta_1, \theta_2) = \left(\frac{2\pi}{k}, 0\right) \right\rangle \qquad \mathbf{Z}_k^2 = \left\langle (\theta_1, \theta_2) = \left(0, \frac{2\pi}{k}\right) \right\rangle.$$

We proceed by analysing these two cases individually.

(a) k_1 even, k_2 odd

In this case ker L is an irreducible representation of H_2. The table of isotropy subgroups and fixed point subspaces is as follows:

Isotropy subgroup	Fixed point subspace	Dimension
$\mathbf{D}_4$	$(0,0)$	0
$\mathbf{Z}_2^1 = \langle \nu_1 \rangle$	$(x_1, 0)$	1
$\mathbf{S}_2 = \langle s_2 \rangle$	(x_1, x_1)	1
$\mathbf{1}$	(x_1, x_2)	2

and the lattice of isotropy subgroups is represented as follows:

$$\begin{tikzcd} & \mathbf{D}_4 & \\ \mathbf{Z}_2^1 \arrow[ur] & & \mathbf{S}_2 \arrow[ul] \\ & \mathbf{1} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thus there are two branches of nonconjugate solutions, one in each of the 1-dimensional fixed point subspaces, and each of them has a conjugate

- $(0, x_2)$ is conjugate to $(x_1, 0)$ by s_2;
- $(x_1, -x_1)$ is conjugate to (x_1, x_1) by ν_1.

The action of H_2 restricted to each of these subspaces is generated by a nontrivial reflection. Thus, all bifurcating branches are pitchforks.

(b) k_1, k_2 **odd**

On the contrary, in this case the group H_2 does not act irreducibly on ker L. An irreducible representation of $\mathbf{Z}_2\dot{+}\mathbf{S}_2$ must be 1-dimensional. In order to make this fact more obvious for this particular representation we change coordinates as follows:

$$X^+ = x_1 + x_2, \qquad X^- = x_1 - x_2.$$

The action of H_2 on the new coordinates is

$$\begin{aligned} \nu_1 &: (X^+, X^-) \mapsto (-X^+, -X^-) \\ s_2 &: (X^+, X^-) \mapsto (X^+, -X^-) \end{aligned}$$

and X^+, X^- are the two irreducible components.

The table of isotropy subgroups and fixed point subspaces is as follows:

Isotropy subgroup	Fixed point subspace	Dimension
$\mathbf{Z}_2\dot{+}\mathbf{S}_2$	$(0,0)$	0
$\mathbf{S}_2^+ = \langle s_2 \rangle$	$(X^+, 0) = (x_1, x_1)$	1
$\mathbf{S}_2^- = \langle s_2\nu_1 \rangle$	$(0, X^-) = (x_1, -x_1)$	1
$\mathbf{1}$	$(X^+, X^-) = (x_1, x_2)$	2

and the lattice of isotropy subgroups is

$$\begin{tikzcd} & \mathbf{Z}_2\dot{+}\mathbf{S}_2 & \\ \mathbf{S}_2^+ \arrow[ur] & & \mathbf{S}_2^- \arrow[ul] \\ & \mathbf{1} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

There are two facts that contrast with the previous case

- (x_1, x_1) and $(x_1, -x_1)$ are nonconjugate;

- $(x_1, 0)$ and $(0, x_2)$ are not fixed point subspaces.

As in the previous case we have that by restricting the action of H_2 to the subspaces (x_1, x_1) and $(x_1, -x_1)$ we find a nontrivial reflection. Thus, the bifurcating branches associated to these subspaces are pitchforks.

Note that H_2 acts as the symmetries of the domain. The assumption that k_1, k_2 are coprime means that we consider a problem with NBC on the (possibly smaller) square obtained by dividing all the original edges by the highest common factor of the original mode numbers. Up to now extra symmetries have been introduced if the original mode numbers are both even. We proceed by showing that the extension property induces more symmetries that remain hidden in the NBC problem and are not contained in H_2.

Recall that the PBC problem has $\mathbf{O}(2)^2 \dot{+} \mathbf{S}_2 = \mathbf{D}_4 \dot{+} \mathbf{T}^2$ as group of symmetries. The group H_2 is the set of elements that leave $\mathrm{Fix}(K)$ invariant. Our aim is to find elements of $\mathbf{T}^2$ that leave certain proper subspaces of $\mathrm{Fix}(K)$ invariant but not the whole $\mathrm{Fix}(K)$. Recall that the action of $\mathbf{T}^2$ on $\ker L = \mathbf{C}^4$ is generated by

$$\begin{array}{ccccc} & z_1^1 & z_2^1 & z_1^2 & z_2^2 \\ \theta_1 & [k_1] & [k_1] & [k_2] & [k_2] \\ \theta_2 & [k_2] & [-k_2] & [k_1] & [-k_1] \end{array}$$

where $[k]$ acts on z by the k-fold action $e^{ik\theta}z$. Recall that the subspace containing the solutions satisfying NBC is

$$\mathrm{Fix}(K) = \left\{(x_1, x_2) | x_1 = \mathrm{Re}(z_j^1), x_2 = \mathrm{Re}(z_j^2) \text{ for } j = 1, 2\right\} = \mathbf{R}^2.$$

The elements $\nu_1, \nu_2 \in H_2$ act on $\mathbf{R}^2$ as the restriction of $\theta_1 = \pi$, $\theta_2 = \pi$ respectively to $\mathrm{Fix}(K)$. There are translations in $\mathbf{T}^2$ that together with K fix proper subspaces of $\mathrm{Fix}(K)$ that are not fixed by any element of H_2. Since k_1, k_2 are both odd, they are nonzero and we can define the translations

$$\chi_{k_1}^1 \equiv (\theta_1, \theta_2) = \left(\frac{2\pi}{k_1}, 0\right) \qquad \chi_{k_2}^2 \equiv (\theta_1, \theta_2) = \left(0, \frac{2\pi}{k_2}\right).$$

Now there is a new fixed point subspace as in the following table together with the associated isotropy subgroup factored by K.

Isotropy subgroup	Fixed point subspace	Dimension
$\mathbf{Z}_{k_1}^1$	$(x_1, 0)$	1

The fixed point subspace $(x_1, 0)$ has the conjugate $(0, x_2)$ by the action of the permutation s_2. Since these fixed point subspaces are 1-dimensional we expect a bifurcating branch associated to each of them. By including the new isotropy subgroup in the

(a) If k_1, k_2 have different parities then A_1 and A_2 must be both even which yields

$$a^1 \equiv a^2 \equiv 0 \pmod 2.$$

(b) If k_1, k_2 have the same parity then a^1 and a^2 achieve their minimum simultaneously when $A_1 = A_2 = 1$ which yields

$$a^1 = a^2 = k$$

where $k = \max(k_1, k_2)$.

We proceed by analysing these two cases separately.

(a) k_1 even, k_2 odd

In this case the restricted $\mathbf{T}^2$-invariants are generated by

$$x_1^2 \qquad \text{and} \qquad x_2^2.$$

Symmetrizing over $\mathbf{S}_2$ we get all the restricted $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-invariants. These are sums of polynomials of the form

$$x_1^{2p} x_2^{2q} + x_2^{2p} x_1^{2q}$$

where without loss of generality we assume that $p \geq q$. This may also be written as

$$\left(x_1^{2(p-q)} + x_2^{2(p-q)}\right)(x_1 x_2)^{2q}. \tag{3.11}$$

Before giving the generators of the symmetrized invariants we state and prove the following:

Lemma 6 *Any polynomial of the form $u^\alpha + v^\alpha$ can be written as sums and products of $u + v$ and uv.*

Proof The proof is by induction and is divided into two distinct cases according to the parities of α.

- If α is odd then

$$u^\alpha + v^\alpha = (u+v)^\alpha - \sum_{j=1}^{\frac{\alpha-1}{2}} \binom{\alpha}{j} (u^{\alpha-j} + v^{\alpha-j})(uv)^j.$$

- If α is even then

$$u^\alpha + v^\alpha = (u+v)^\alpha - \sum_{j=1}^{\frac{\alpha}{2}-1} \binom{\alpha}{j} (u^{\alpha-j} + v^{\alpha-j})(uv)^j - \binom{\alpha}{\frac{\alpha}{2}} (uv)^{\frac{\alpha}{2}}.$$

By continuing this procedure we eventually get sums and products of the basic polynomials

$$u + v \qquad \text{and} \qquad uv.$$

$\square$

Proposition 5 *The symmetrization over $\mathbf{S}_2$ of the polynomials generated by x_1^2, x_2^2 is generated by*

$$x_1^2 + x_2^2 \qquad \textit{and} \qquad x_1^2 x_2^2.$$

Proof Recall that the symmetrization of a polynomial generated by x_1^2, x_2^2 is written as sums of polynomials of the form (3.11). By making $u = x_1^2$, $v = x_2^2$, lemma 6 says that the symmetrized invariants are generated by $x_1^2 + x_2^2$ and $x_1^2 x_2^2$. □

Proposition 6 *The restricted equivariants are generated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix}.$$

Proof By theorem 7 in chapter 2 we have that the restriction of the $\mathbf{T}^2$-equivariants to $\mathrm{Fix}(K)$ is the modulo generated by

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

over the ring of restricted $\mathbf{T}^2$-invariants, which is generated by x_1^2, x_2^2. Symmetrizing over $\mathbf{S}_2$ we get all the restricted $\mathbf{D}_4\dot{+}\mathbf{T}^2$-equivariants. These are sums of polynomial mappings of the form

$$\begin{pmatrix} x_1^{2p+1} x_2^{2q} \\ x_2^{2p+1} x_1^{2q} \end{pmatrix}. \tag{3.12}$$

Now we can see that the mappings

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix}$$

are of the required form. We claim that the converse is also true: any mapping of the form (3.12) can be written as

$$a \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix},$$

where a, b are polynomial functions of $x_1^2 + x_2^2$ and $x_1^2 x_2^2$. To prove the claim we have to show that given any pair of nonnegative integers p, q, there exists a pair of polynomial functions a, b such that

$$\begin{pmatrix} x_1^{2p+1} x_2^{2q} \\ x_2^{2p+1} x_1^{2q} \end{pmatrix} = a(x_1^2 + x_2^2, x_1^2 x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b(x_1^2 + x_2^2, x_1^2 x_2^2) \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix}. \tag{3.13}$$

We may assume without loss of generality that $p \geq q$, in which case by factoring out $(x_1 x_2)^{2q}$ from (3.13) we get

$$\begin{pmatrix} x_1^{2(p-q)+1} \\ x_2^{2(p-q)+1} \end{pmatrix} = \tilde{a}(x_1^2 + x_2^2, x_1^2 x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tilde{b}(x_1^2 + x_2^2, x_1^2 x_2^2) \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix}, \tag{3.14}$$

where

$$\begin{aligned} \tilde{a} &= (x_1x_2)^{-2q}a \\ \tilde{b} &= (x_1x_2)^{-2q}b. \end{aligned}$$

By denoting $r = p - q + 1$ we have that the left hand side of (3.14) is a scalar multiple of a gradient

$$\begin{pmatrix} x_1^{2r-1} \\ x_2^{2r-1} \end{pmatrix} = \frac{1}{2r}\nabla(x_1^{2r} + x_2^{2r}).$$

By lemma 6, the polynomial $x_1^{2r} + x_2^{2r}$ can be written as sums and products of $x_1^2 + x_2^2$ and $x_1^2x_2^2$, which gradient is a combination of the basic gradients

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\nabla(x_1^2 + x_2^2) \qquad \text{and} \qquad \begin{pmatrix} x_1x_2^2 \\ x_2x_1^2 \end{pmatrix} = \frac{1}{2}\nabla(x_1^2x_2^2)$$

over $x_1^2 + x_2^2$ and $x_1^2x_2^2$. This is what we wanted. □

Theorem 14 *Assume that $\mathcal{P}(u) = 0$ satisfying NBC on the square $[0,\pi] \times [0,\pi]$ undergoes a single mode bifurcation with mode numbers $(k_1, k_2) \in \mathbb{N}^2$ when the parameter λ crosses zero. Then, if k_1, k_2 have different parities, the bifurcation equations on ker L are*

$$\begin{aligned} f_1(x_1,x_2,\lambda) = a(I_1,I_2,\lambda)x_1 + b(I_1,I_2,\lambda)x_1x_2^2 = 0 \\ f_2(x_1,x_2,\lambda) = a(I_1,I_2,\lambda)x_2 + b(I_1,I_2,\lambda)x_2x_1^2 = 0, \end{aligned} \tag{3.15}$$

where $I_1 = x_1^2 + x_2^2$ and $I_2 = x_1^2x_2^2$ generate the invariants.

Proof Immediate from propositions 5 and 6. □

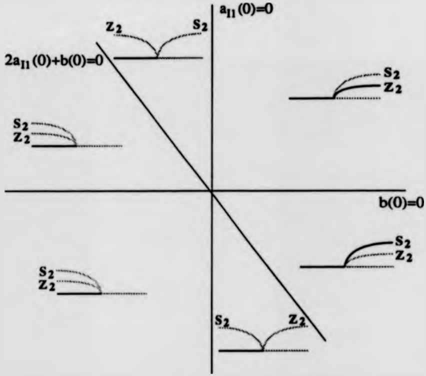
The branching equations and stability for the invariant system

$$\begin{aligned} \dot{x}_1 + f_1(x_1,x_2,\lambda) = 0 \\ \dot{x}_2 + f_2(x_1,x_2,\lambda) = 0 \end{aligned} \tag{3.16}$$

where f_1, f_2 are as in (3.15) are in Golubitsky *et al.* [11] with different coordinates. In the coordinates adopted here the results are as follows:

Isotropy subgroup	Branching equations	Eigenvalues
$\mathbf{Z}_2$	$x_2 = 0$	$a_{I_1}(0)$
	$a_{I_1}(0)x_1^2 + a_\lambda(0)\lambda = 0$	$b(0)$
$\mathbf{S}_2$	$x_2 = x_1$	$2a_{I_1}(0) + b(0)$
	$(2a_{I_1}(0) + b(0))x_1^2 + a_\lambda(0)\lambda = 0$	$-b(0)$

Assuming that $a_\lambda(0) < 0$ the bifurcation diagram depends on the coefficients $a_{l_1}(0)$ and $b(0)$ as follows:



Note that the branches found here are exactly the ones predicted by the group theory in section 3.5.2. They all live in 1-dimensional fixed point subspaces.

(b) k_1, k_2 odd

In this case the restricted $\mathbf{T}^2$-invariants are generated by

$$x_1^2, \qquad x_2^2 \qquad \text{and} \qquad x_1^k x_2^k.$$

Symmetrizing over $\mathbf{S}_2$, the invariants are sums of polynomials of the form

$$\left(x_1^{2p}x_2^{2q} + x_2^{2p}x_1^{2q}\right)(x_1x_2)^{kc} = \left(x_1^{2(p-q)} + x_2^{2(p-q)}\right)(x_1x_2)^{2q}(x_1x_2)^{kc}. \qquad (3.17)$$

and these are generated as follows:

Proposition 7 *The symmetrization over $\mathbf{S}_2$ of the polynomials generated by x_1^2, x_2^2 and $x_1^k x_2^k$ is generated by*

$$x_1^2 + x_2^2, \qquad x_1^2 x_2^2 \qquad \textit{and} \qquad x_1^k x_2^k.$$

Proof This comes immediately by making $u = x_1^2$, $v = x_2^2$ in lemma 6 together with the fact that the symmetrized invariants are sums of polynomials of the form (3.17). □

Proposition 8 *The restricted equivariants are generated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 x_2^2 \\ x_2 x_1^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1^{k-1} x_2^k \\ x_2^{k-1} x_1^k \end{pmatrix}.$$

Proof Analogous to that of proposition 6. □

Theorem 15 *Assume that $\mathcal{P}(u) = 0$ satisfying NBC on the square $[0,\pi] \times [0,\pi]$ undergoes a single mode bifurcation with mode numbers $(k_1, k_2) \in \mathbb{N}^2$ when the parameter λ crosses zero. Then, if k_1, k_2 are both odd, the bifurcation equations on ker L are*

$$\begin{aligned} f_1(x_1, x_2, \lambda) &= a(I_1, I_2, I_3, \lambda)x_1 + b(I_1, I_2, I_3, \lambda)x_1 x_2^2 + c(I_1, I_2, I_3, \lambda)x_1^{k-1}x_2^k = 0 \\ f_2(x_1, x_2, \lambda) &= a(I_1, I_2, I_3, \lambda)x_2 + b(I_1, I_2, I_3, \lambda)x_2 x_1^2 + c(I_1, I_2, I_3, \lambda)x_2^{k-1}x_1^k = 0 \end{aligned} \tag{3.18}$$

where $k = \max(k_1, k_2)$ and $I_1 = x_1^2 + x_2^2$, $I_2 = x_1^2 x_2^2$ and $I_3 = x_1^k x_2^k$ generate the invariants.

Proof Immediate from propositions 7 and 8. □

In this case the form of the invariant system is

$$\begin{aligned} \dot{x}_1 + f_1(I_1, I_2, I_3, \lambda) &= 0 \\ \dot{x}_2 + f_2(I_1, I_2, I_3, \lambda) &= 0 \end{aligned} \tag{3.19}$$

where f_1, f_2 are as (3.18). Given that $k \geq 3$ the truncation at third order is the generic $\mathbf{D}_4$-invariant system. There are terms breaking this symmetry to $\mathbf{Z}_2 \dot{+} \mathbf{S}_2$ but their order on both x_1 and x_2 is high enough to have only the effect of bending the branches given in the previous case (k_1 even, k_2 odd) where the system was $\mathbf{D}_4$-invariant. Again there are no more branches apart from the ones associated to the 1-dimensional fixed point subspaces found in section 3.5.2.

3.6 3-Dimensional Domain

This section is essentially group theoretical. The generic bifurcation equations will not be given explicitly because of the combinatorial complexity in finding a minimal set of generators for the restricted invariants. There are three cases to consider.

3.6.1 $k_1 = k_2 = k_3$

In this case ker L is 1-dimensional and the action of the normalizer quotient H_3 is generated as in section 3.4. The reflections ν_1, ν_2, ν_3 act in the same way (nontrivially) and the permutations s_j, for $1 \leq j \leq 6$ act trivially. Thus, as in the case $n = 1$, the generic bifurcation from $x = 0$ is a pitchfork.

3.6.2 $k_1 \neq k_2 = k_3$

In this case ker L is 2-dimensional. Any element of the symmetric group $\mathbf{S}_3$ acts as one of the following permutations

$$s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The reflections ν_2, ν_3 have the same action on $\mathbb{R}^2$. Therefore, the action of H_3 is generated as in section 3.5 and the analysis for the case $n = 2$ applies directly here.

3.6.3 $k_1 \neq k_2 \neq k_3, k_1 \neq k_3$: Group Theory

The symmetric group $\mathbf{S}_3$ contains the 6 elements

$$s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad s_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
$$s_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad s_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad s_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

and s_2, s_3 are the generators. The action of H_3 is nonminimally generated by

	x_1	x_2	x_3	x_4	x_5	x_6
ν_1	$(-1)^{k_1}x_1$	$(-1)^{k_2}x_2$	$(-1)^{k_3}x_3$	$(-1)^{k_2}x_4$	$(-1)^{k_3}x_5$	$(-1)^{k_1}x_6$
ν_2	$(-1)^{k_2}x_1$	$(-1)^{k_1}x_2$	$(-1)^{k_2}x_3$	$(-1)^{k_3}x_4$	$(-1)^{k_1}x_5$	$(-1)^{k_3}x_6$
ν_3	$(-1)^{k_3}x_1$	$(-1)^{k_3}x_2$	$(-1)^{k_1}x_3$	$(-1)^{k_1}x_4$	$(-1)^{k_2}x_5$	$(-1)^{k_2}x_6$
s_1	x_1	x_2	x_3	x_4	x_5	x_6
s_2	x_2	x_1	x_4	x_3	x_6	x_5
s_3	x_3	x_5	x_1	x_6	x_2	x_4
s_4	x_4	x_6	x_2	x_5	x_1	x_3
s_5	x_5	x_3	x_6	x_1	x_4	x_2
s_6	x_6	x_4	x_5	x_2	x_3	x_1

and ν_1, s_2, s_3 form a minimal set of generators. By theorem 13 we have that H_3 is isomorphic to

(a) $\mathbf{O}$ if two of k_1, k_2, k_3 are even and the other is odd;

(b) $(\mathbf{Z}_2)^2 \dot{+} \mathbf{S}_3$ if one of k_1, k_2, k_3 is even and the others are odd;

(c) $\mathbf{Z}_2 \dot{+} \mathbf{S}_3$ if k_1, k_2, k_3 are all odd.

In any of these cases an irreducible representation of H_3 cannot have dimension greater than 3. Thus $\mathbb{R}^6$ is necessarily reducible. In order to emphasize this fact we introduce the coordinates

$$X_1^{\pm} = x_1 \pm x_6 \qquad X_2^{\pm} = x_2 \pm x_5 \qquad X_3^{\pm} = x_3 \pm x_4.$$

As we will see case by case, the two components X^+ and X^- are not mixed by the action of H_3 and they may be irreducible or not depending on the parities of the mode numbers.

As in sections 3.4 and 3.5 the normalizer quotient H_3 introduces extra symmetries in the NBC problem if and only if all the mode numbers are even. This possibility has already been excluded by assuming that k_1, k_2, k_3 are coprime, but the common factors have to be taken into account when interpreting the results. By analogy with section 3.5 we will find more symmetries of the periodic extension that remain hidden in the NBC problem and are not contained in H_3.

Recall that the group of symmetries of the PBC problem is $\mathbf{O}(2)^3 \dot{+} \mathbf{S}_3 = \mathbf{O} \dot{+} \mathbf{T}^3$ and K is the subgroup of reflections, isomorphic to $(\mathbf{Z}_2)^3$, such that $K \dot{+} \mathbf{T}^3 = \mathbf{O}(2)^3$. We denoted H_3 as the subgroup that leaves $\mathrm{Fix}(K)$ invariant factored by the kernel of its action on this fixed point subspace. We want elements of $\mathbf{T}^3$ that leave invariant proper subspaces $\mathrm{Fix}(K')$ but not the whole $\mathrm{Fix}(K)$. The action of $\mathbf{T}^3$ on $\ker L = \mathbf{C}^{24}$ is generated by

$$\begin{array}{ccccccccc}
 & z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_1^2 & z_2^2 & z_3^2 & z_4^2 \\
\theta_1 & [k_1] & [k_1] & [k_1] & [k_1] & [k_2] & [k_2] & [k_2] & [k_2] \\
\theta_2 & [k_2] & [-k_2] & [k_2] & [-k_2] & [k_1] & [-k_1] & [k_1] & [-k_1] \\
\theta_3 & [k_3] & [k_3] & [-k_3] & [-k_3] & [k_3] & [k_3] & [-k_3] & [-k_3] \\
 & & & & & & & & \\
 & z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_1^4 & z_2^4 & z_3^4 & z_4^4 \\
\theta_1 & [k_3] & [k_3] & [k_3] & [k_3] & [k_2] & [k_2] & [k_2] & [k_2] \\
\theta_2 & [k_2] & [-k_2] & [k_2] & [-k_2] & [k_3] & [-k_3] & [k_3] & [-k_3] \\
\theta_3 & [k_1] & [k_1] & [-k_1] & [-k_1] & [k_1] & [k_1] & [-k_1] & [-k_1] \\
 & & & & & & & & \\
 & z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_1^6 & z_2^6 & z_3^6 & z_4^6 \\
\theta_1 & [k_3] & [k_3] & [k_3] & [k_3] & [k_1] & [k_1] & [k_1] & [k_1] \\
\theta_2 & [k_1] & [-k_1] & [k_1] & [-k_1] & [k_3] & [-k_3] & [k_3] & [-k_3] \\
\theta_3 & [k_2] & [k_2] & [-k_2] & [-k_2] & [k_2] & [k_2] & [-k_2] & [-k_2]
\end{array}$$

where again $[k]$ acts on z by k-fold $e^{ik\theta}z$ and K acts on each block $z^j \in \mathbf{C}^4$ as

$$\begin{array}{ccccc}
 & z_1^j & z_2^j & z_3^j & z_4^j \\
\kappa_1 & \bar{z}_4^j & \bar{z}_3^j & \bar{z}_2^j & \bar{z}_1^j \\
\kappa_2 & z_2^j & z_1^j & z_4^j & z_3^j \\
\kappa_3 & z_3^j & z_4^j & z_1^j & z_2^j
\end{array}$$

for $1 \leq j \leq 6$. Now the subspace containing the solutions satisfying NBC is

$$\mathrm{Fix}(K) = \left\{(x_1, \ldots, x_6) | x_j = \mathrm{Re}(z_1^j) = \mathrm{Re}(z_2^j) = \mathrm{Re}(z_3^j) = \mathrm{Re}(z_4^j)\right\} = \mathbf{R}^6.$$

The subgroup of H_3 that is contained in $\mathbf{T}^3$ is generated by the translations

$$\begin{aligned}
\nu_1 &\equiv (\theta_1, \theta_2, \theta_3) = (\pi, 0, 0) \\
\nu_2 &\equiv (\theta_1, \theta_2, \theta_3) = (0, \pi, 0) \\
\nu_3 &\equiv (\theta_1, \theta_2, \theta_3) = (0, 0, \pi).
\end{aligned}$$

The translations that when added to K fix proper subspaces of $\mathrm{Fix}(K)$ are generated by $\chi^j_{k_1}, \chi^j_{k_2}, \chi^j_{k_3}, \chi^j_{h_1}, \chi^j_{h_2}, \chi^j_{h_3}$ for $1 \leq j \leq 3$ where

$$\chi^1_k \equiv (\theta_1, \theta_2, \theta_3) = \left(\frac{2\pi}{k}, 0, 0\right)$$

$$\chi^2_k \equiv (\theta_1, \theta_2, \theta_3) = \left(0, \frac{2\pi}{k}, 0\right)$$

$$\chi^3_k \equiv (\theta_1, \theta_2, \theta_3) = \left(0, 0, \frac{2\pi}{k}\right)$$

and

$$h_1 = \mathrm{hcf}(k_1, k_2) \qquad h_2 = \mathrm{hcf}(k_2, k_3) \qquad h_3 = \mathrm{hcf}(k_1, k_3).$$

Again we denote

$$\mathbf{Z}^1_k = \langle \chi^1_k \rangle \qquad \mathbf{Z}^2_k = \langle \chi^2_k \rangle \qquad \mathbf{Z}^3_k = \langle \chi^3_k \rangle.$$

Depending on the common factors, h_1, h_2, h_3, these subgroups together with K may imply the existence of more fixed point subspaces. These together with the corresponding constraints are as follows:

Isotropy subgroup	Fixed point subspace	Dim	Constraint
$\mathbf{Z}^1_{k_1}$	$(X_1^+, 0, 0, X_1^-, 0, 0)$	2	$k_1 > h_1, h_3$
$\mathbf{Z}^1_{k_2}$	$(0, X_2^+, X_3^+, 0, X_2^+, -X_3^+)$	2	$k_2 > h_1, h_2$
$\mathbf{Z}^1_{k_3}$	$(0, X_2^+, X_3^+, 0, -X_2^+, X_3^+)$	2	$k_3 > h_2, h_3$
$\mathbf{Z}^1_{h_1}$	$(X_1^+, X_2^+, X_3^+, X_1^-, X_2^+, -X_3^+)$	4	$h_1 > 1$
$\mathbf{Z}^1_{h_2}$	$(0, X_2^+, X_3^+, 0, X_2^-, X_3^-)$	4	$h_2 > 1$
$\mathbf{Z}^1_{h_3}$	$(X_1^+, X_2^+, X_3^+, X_1^-, -X_2^+, X_3^+)$	4	$h_3 > 1$

Now we have that

$$\chi^2_k = s_2 \chi^1_k s_2^{-1} \qquad \text{and} \qquad \chi^3_k = s_3 \chi^1_k s_3^{-1}.$$

Thus, the subspaces fixed by χ^2_k and χ^3_k are

$$\mathrm{Fix}(\chi^2_k) = s_2 \mathrm{Fix}(\chi^1_k) \qquad \text{and} \qquad \mathrm{Fix}(\chi^3_k) = s_3 \mathrm{Fix}(\chi^1_k).$$

Note that the three conditions $h_1, h_2, h_3 > 1$ imply that $k_1 > h_1, h_3$, $k_2 > h_1, h_2$ and $k_3 > h_2, h_3$. So for the existence of all the fixed point subspaces in the table above it is enough to assume $h_1, h_2, h_3 > 1$.

(a) k_1 odd, k_2, k_3 even

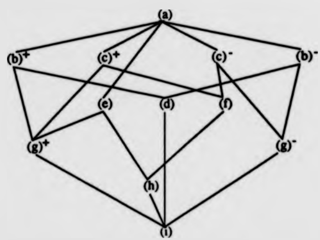
In this case the group H_3 is isomorphic to $\mathbf{O}$ and its action is generated by

	X_1^+	X_2^+	X_3^+	X_1^-	X_2^-	X_3^-
ν_1	$-X_1^+$	X_2^+	X_3^+	$-X_1^-$	X_2^-	X_3^-
ν_2	X_1^+	$-X_2^+$	X_3^+	X_1^-	$-X_2^-$	X_3^-
ν_3	X_1^+	X_2^+	$-X_3^+$	X_1^-	X_2^-	$-X_3^-$
s_1	X_1^+	X_2^+	X_3^+	X_1^-	X_2^-	X_3^-
s_2	X_2^+	X_1^+	X_3^+	X_2^-	X_1^-	$-X_3^-$
s_3	X_3^+	X_2^+	X_1^+	X_3^-	$-X_2^-$	X_1^-
s_4	X_3^+	X_1^+	X_2^+	$-X_3^-$	$-X_1^-$	X_2^-
s_5	X_2^+	X_3^+	X_1^+	$-X_2^-$	X_3^-	$-X_1^-$
s_6	X_1^+	X_3^+	X_2^+	$-X_1^-$	$-X_3^-$	$-X_2^-$

and X^+, X^- are the two irreducible components. By denoting $\nu = \nu_1\nu_2\nu_3$ the isotropy subgroups and fixed point subspaces are

Label	Isotropy subgroup	Fixed point subspace	Dimension
(a)	$\mathbf{O} = \langle \nu_1, s_2, s_3 \rangle$	$(0,0,0,0,0,0)$	0
(b)$^+$	$\mathbf{S}_3^+ = \langle s_2, s_3 \rangle$	$(X_1^+, X_1^+, X_1^+, 0, 0, 0)$	1
(b)$^-$	$\mathbf{S}_3^- = \langle s_2\nu, s_3\nu \rangle$	$(0,0,0, X_1^-, -X_1^-, -X_1^-)$	1
(c)$^+$	$\mathbf{D}_4^+ = \langle \nu_1, s_2 \rangle$	$(0,0,X_3^+,0,0,0)$	1
(c)$^-$	$\mathbf{D}_4^- = \langle \nu_1, s_2\nu_3 \rangle$	$(0,0,0,0,0,X_3^-)$	1
(d)	$\mathbf{Z}_3 = \langle s_4 \rangle$	$(X_1^+, X_1^+, X_1^+, X_1^-, -X_1^-, -X_1^-)$	2
(e)	$\mathbf{Z}_2^1 \oplus \mathbf{S}_2 = \langle \nu_1, s_6 \rangle$	$(0, X_2^+, X_2^+, 0, X_2^-, -X_2^-)$	2
(f)	$\mathbf{Z}_2^1 \oplus \mathbf{Z}_2^2 = \langle \nu_1, \nu_2 \rangle$	$(0,0,X_3^+,0,0,X_3^-)$	2
(g)$^+$	$\mathbf{S}_2^+ = \langle s_6 \rangle$	$(X_1^+, X_2^+, X_2^+, 0, X_2^-, -X_2^-)$	3
(g)$^-$	$\mathbf{S}_2^- = \langle s_6\nu_1 \rangle$	$(0, X_2^+, X_2^+, X_1^-, X_2^-, -X_2^-)$	3
(h)	$\mathbf{Z}_2^1 = \langle \nu_1 \rangle$	$(0, X_2^+, X_3^+, 0, X_2^-, X_3^-)$	4
(i)	$\mathbf{1}$	$(X_1^+, X_2^+, X_3^+, X_1^-, X_2^-, X_3^-)$	6

and the lattice of isotropy subgroups is as follows:



Up to now we know, by the equivariant branching lemma, the existence of branches in the subspaces labeled as (b)$^\pm$, (c)$^\pm$ and the respective conjugates by the action of H_3. These are in the following subspaces

$(b)^+$ $(X, X, X, 0, 0, 0)$
$(-X, X, X, 0, 0, 0)$ conjugate by ν_1
$(X, -X, X, 0, 0, 0)$ conjugate by ν_2
$(X, X, -X, 0, 0, 0)$ conjugate by ν_3

$(b)^-$ $(0, 0, 0, X, -X, -X)$
$(0, 0, 0, -X, -X, -X)$ conjugate by ν_1
$(0, 0, 0, X, X, -X)$ conjugate by ν_2
$(0, 0, 0, X, -X, X)$ conjugate by ν_3

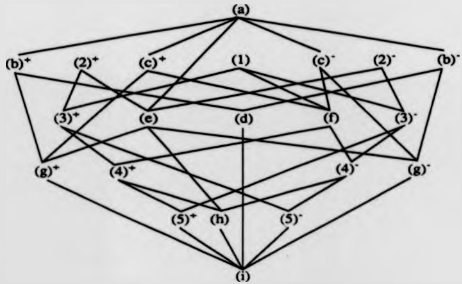
$(c)^+$ $(0, 0, X, 0, 0, 0)$
$(X, 0, 0, 0, 0, 0)$ conjugate by s_4
$(0, X, 0, 0, 0, 0)$ conjugate by s_5

$(c)^-$ $(0, 0, 0, 0, 0, X)$
$(0, 0, 0, X, 0, 0)$ conjugate by s_4
$(0, 0, 0, 0, X, 0)$ conjugate by s_5

By including the subgroups of $\mathbf{T}^3$ that fix proper subspaces of $\mathrm{Fix}(K)$ we get a much richer table of isotropy subgroups. By assuming that all the constraints imposed on h_1, h_2, h_3 are valid we have the new fixed point subspaces as follows:

Label	Isotropy subgroup	Fixed point subspace	Dimension
(1)	$\mathbf{Z}^1_{h_1} \oplus \mathbf{Z}^2_{h_2} \oplus \mathbf{Z}^3_{h_1}$	$(0, 0, X_3^+, 0, 0, X_3^+)$	1
$(2)^+$	$\mathbf{Z}^1_{h_3} \oplus ((\mathbf{Z}^2_{h_1} \oplus \mathbf{Z}^3_{h_1}) \dot{+} \langle s_6 \rangle)$	$(0, X_2^+, X_2^+, 0, -X_2^+, X_2^+)$	1
$(2)^-$	$\mathbf{Z}^1_{h_3} \oplus ((\mathbf{Z}^2_{h_3} \oplus \mathbf{Z}^3_{h_3}) \dot{+} \langle s_6 \rangle)$	$(0, X_2^+, X_2^+, 0, X_2^+, -X_2^+)$	1
$(3)^+$	$\mathbf{Z}^1_{h_3} \oplus \mathbf{Z}^2_{h_1} \oplus \mathbf{Z}^3_{h_1}$	$(0, X_2^+, X_3^+, 0, -X_2^+, X_3^+)$	2
$(3)^-$	$\mathbf{Z}^1_{h_3} \oplus \mathbf{Z}^2_{h_3} \oplus \mathbf{Z}^3_{h_3}$	$(0, X_2^+, X_3^+, 0, X_2^+, -X_3^+)$	2
$(4)^+$	$\mathbf{Z}^1_{h_2} \oplus \mathbf{Z}^3_{h_1}$	$(0, X_2^+, X_3^+, 0, -X_2^+, X_3^-)$	3
$(4)^-$	$\mathbf{Z}^1_{h_2} \oplus \mathbf{Z}^3_{h_3}$	$(0, X_2^+, X_3^+, 0, X_2^+, X_3^-)$	3
$(5)^+$	$\mathbf{Z}^3_{h_1}$	$(X_1^+, X_2^+, X_3^+, -X_1^+, -X_2^+, X_3^-)$	4
$(5)^-$	$\mathbf{Z}^3_{h_3}$	$(X_1^+, X_2^+, X_3^+, X_1^+, X_2^+, X_3^-)$	4

By including these subgroups in the isotropy lattice we get



Again by the equivariant branching lemma we prove the existence of three more non-conjugate branches. These and the respective conjugates are in the subspaces

(1) $(0,0,X,0,0,X)$
$(0,0,X,0,0,-X)$ conjugate by s_2
$(X,0,0,X,0,0)$ conjugate by s_3
$(X,0,0,-X,0,0)$ conjugate by s_4
$(0,X,0,0,X,0)$ conjugate by s_5
$(0,X,0,0,-X,0)$ conjugate by s_6

$(2)^+$ $(0,X,X,0,-X,X)$
$(X,0,X,-X,0,-X)$ conjugate by s_2
$(X,X,0,X,X,0)$ conjugate by s_3
$(0,-X,X,0,X,X)$ conjugate by ν_2
$(-X,0,X,X,0,-X)$ conjugate by $\nu_2 s_2$
$(X,-X,0,X,-X,0)$ conjugate by $\nu_2 s_3$

$(2)^-$ $(0,X,X,0,X,-X)$
$(X,0,X,X,0,-X)$ conjugate by s_2
$(X,X,0,-X,-X,0)$ conjugate by s_3
$(0,-X,X,0,-X,-X)$ conjugate by ν_2
$(-X,0,X,-X,0,X)$ conjugate by $\nu_2 s_2$
$(X,-X,0,-X,X,0)$ conjugate by $\nu_2 s_3$

By the reflection ν all branches are pitchforks.

(b) k_1 even, k_2, k_3 odd

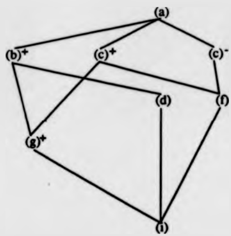
Now the group H_3 is isomorphic to $(\mathbf{Z}_2)^2 \dot{+} \mathbf{S}_3$ and its action is generated by

$$\begin{array}{ccccccc}
 & X_1^+ & X_2^+ & X_3^+ & X_1^- & X_2^- & X_3^- \\
\nu_1 & X_1^+ & -X_2^+ & -X_3^+ & X_1^- & -X_2^- & -X_3^- \\
\nu_2 & -X_1^+ & X_2^+ & -X_3^+ & -X_1^- & X_2^- & -X_3^- \\
\nu_3 & -X_1^+ & -X_2^+ & X_3^+ & -X_1^- & -X_2^- & X_3^-
\end{array}$$

and the s_j acting as before. Now X^+, X^- are the two irreducible components. The isotropy subgroups and fixed point subspaces are

Label	Isotropy subgroup	Fixed point subspace	Dimension
(a)	$\mathbf{Z}_2 \dot{+} \mathbf{S}_3 = \langle \nu_1, s_2, s_3 \rangle$	$(0,0,0,0,0,0)$	0
$(b)^+$	$\mathbf{S}_3 = \langle s_2, s_3 \rangle$	$(X_1^+, X_1^+, X_1^+, 0, 0, 0)$	1
$(c)^+$	$\mathbf{Z}_2^3 \oplus \mathbf{S}_2^+ = \langle \nu_3, s_2 \rangle$	$(0, 0, X_3^+, 0, 0, 0)$	1
$(c)^-$	$\mathbf{Z}_2^3 \oplus \mathbf{S}_2^- = \langle \nu_3, s_2\nu_1 \rangle$	$(0,0,0,0,0,X_3^-)$	1
(d)	$\mathbf{Z}_3 = \langle s_4 \rangle$	$(X_1^+, X_1^+, X_1^+, X_1^-, -X_1^-, -X_1^-)$	2
(f)	$\mathbf{Z}_2^3 = \langle \nu_3 \rangle$	$(0, 0, X_3^+, 0, 0, X_3^-)$	2
$(g)^+$	$\mathbf{S}_2 = \langle s_6 \rangle$	$(X_1^+, X_2^+, X_2^+, 0, X_2^-, -X_2^-)$	3
(i)	1	$(X_1^+, X_2^+, X_3^+, X_1^-, X_2^-, X_3^-)$	6

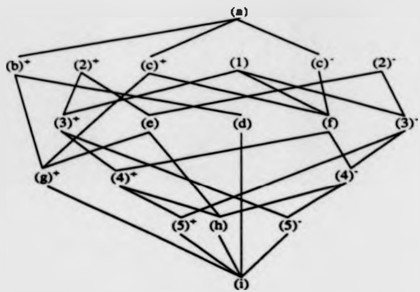
and the lattice of isotropy subgroups is as follows:



By comparing with the previous case (k_1 odd, k_2, k_3 even) where $H_3 = \mathbf{O}$ we note that $(b)^-$, (e), $(g)^-$ and (h) are no longer fixed point subspaces. In this case the equivariant branching lemma guarantees bifurcating branches in the subspaces $(b)^+$ and $(c)^\pm$. In contrast to with the previous case, the subspace $(b)^-$ is no longer a 1-dimensional fixed point subspace and the branch in $(b)^+$ is not a pitchfork. By taking into account all the symmetries that fix some proper subspace of $\mathrm{Fix}(K)$ we recover some of them, but not all. This contrasts with the 2-dimensional domain where in the case $H_2 = \mathbf{Z}_2 \dot{+} \mathbf{S}_2$ we recover by this procedure all the subspaces that are fixed by subgroups of $\mathbf{D}_4$. Under the assumption $h_1, h_2, h_3 > 1$ the new fixed point subspaces are as follows:

Label	Isotropy subgroup	Fixed point subspace	Dimension
(1)	$\mathbf{Z}^1_{h_1}\oplus\mathbf{Z}^2_{h_2}\oplus\mathbf{Z}^3_{h_3}$	$(0,0,X_3^+,0,0,X_3^+)$	1
$(2)^+$	$\mathbf{Z}^1_{h_3}\oplus((\mathbf{Z}^2_{h_1}\oplus\mathbf{Z}^3_{h_1})\dot{+}(s_6))$	$(0,X_2^+,X_2^+,0,-X_2^+,X_2^+)$	1
$(2)^-$	$\mathbf{Z}^1_{h_3}\oplus((\mathbf{Z}^2_{h_2}\oplus\mathbf{Z}^3_{h_3})\dot{+}(s_6))$	$(0,X_2^+,X_2^+,0,X_2^+,-X_2^+)$	1
(e)	$\mathbf{Z}^1_{h_2}\oplus(s_6)$	$(0,X_2^+,X_2^+,0,X_2^-,-X_2^-)$	2
$(3)^+$	$\mathbf{Z}^1_{h_3}\oplus\mathbf{Z}^2_{h_1}\oplus\mathbf{Z}^3_{h_1}$	$(0,X_2^+,X_3^+,0,-X_2^+,X_3^+)$	2
$(3)^-$	$\mathbf{Z}^1_{h_2}\oplus\mathbf{Z}^2_{h_3}\oplus\mathbf{Z}^3_{h_3}$	$(0,X_2^+,X_3^+,0,X_2^+,-X_3^+)$	2
$(4)^+$	$\mathbf{Z}^1_{h_2}\oplus\mathbf{Z}^3_{h_1}$	$(0,X_2^+,X_3^+,0,-X_2^+,X_3^-)$	3
$(4)^-$	$\mathbf{Z}^1_{h_2}\oplus\mathbf{Z}^3_{h_3}$	$(0,X_2^+,X_3^+,0,X_2^+,X_3^-)$	3
(h)	$\mathbf{Z}^1_{h_2}$	$(0,X_2^+,X_3^+,0,X_2^-,X_3^-)$	4
$(5)^+$	$\mathbf{Z}^3_{h_1}$	$(X_1^+,X_2^+,X_3^+,-X_1^+,-X_2^+,X_3^-)$	4
$(5)^-$	$\mathbf{Z}^3_{h_3}$	$(X_1^+,X_2^+,X_3^+,X_1^+,X_2^+,X_3^-)$	4

Comparing with the case k_1 odd, k_2,k_3 even there are still some fixed point subspaces that have not been found here, namely $(b)^-$ and $(g)^-$. By including the new subgroups in the isotropy lattice we get



(c) k_1,k_2,k_3 odd

The group H_3 is isomorphic to $\mathbf{Z}_2\dot{+}\mathbf{S}_3$ and its action is generated by

$$\begin{array}{ccccccc}
 & X_1^+ & X_2^+ & X_3^+ & X_1^- & X_2^- & X_3^- \\
\nu_1 & -X_1^+ & -X_2^+ & -X_3^+ & -X_1^- & -X_2^- & -X_3^- \\
\nu_2 & -X_1^+ & -X_2^+ & -X_3^+ & -X_1^- & -X_2^- & -X_3^- \\
\nu_3 & -X_1^+ & -X_2^+ & -X_3^+ & -X_1^- & -X_2^- & -X_3^-
\end{array}$$

and the s_j acting as before. The group acts diagonally on $X=(X^+,X^-)$, but X^+ and X^- are reducible. In order to find the irreducible components we change coordinates once more by

$$
\begin{aligned}
u^+ &= X_1^+ + X_2^+ + X_3^+ \\
v^+ &= X_1^+ + e^{\frac{2\pi i}{3}} X_2^+ + e^{-\frac{2\pi i}{3}} X_3^+ \\
u^- &= X_1^- - X_2^- - X_3^- \\
v^- &= X_1^- + e^{-\frac{\pi i}{3}} X_2^- + e^{\frac{\pi i}{3}} X_3^-
\end{aligned}
$$

Now $\mathbf{Z}_2 \oplus \mathbf{D}_3 = \mathbf{Z}_2 \dot{+} \mathbf{S}_3$ acts diagonally on $\mathbb{R} \times \mathbb{C} \times \mathbb{R} \times \mathbb{C}$ as follows:

$$
\begin{array}{ccccc}
 & u^+ & v^+ & u^- & v^- \\
\nu_1 & -u^+ & e^{\pi i} v^+ & -u^- & e^{\pi i} v^- \\
s_5 & u^+ & e^{\frac{2\pi i}{3}} v^+ & u^- & e^{\frac{2\pi i}{3}} v^- \\
s_6 & u^+ & \overline{v}^+ & -u^- & -\overline{v}^-
\end{array}
$$

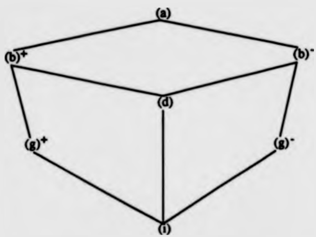
and each of u^+, v^+, u^-, v^- is irreducible. The group $\mathbf{Z}_2 \oplus \mathbf{D}_3$ is isomorphic to $\mathbf{D}_6$ and a minimal set of generators acts as follows:

$$
\begin{array}{ccccc}
 & u^+ & v^+ & u^- & v^- \\
\nu_1 s_4 & -u^+ & e^{\frac{\pi i}{3}} v^+ & -u^- & e^{\frac{\pi i}{3}} v^- \\
s_6 & u^+ & \overline{v}^+ & -u^- & -\overline{v}^-
\end{array}
$$

In order to compare this case with the previous ones it is more convenient to give the fixed point subspaces in the coordinates X^+, X^-.

Label	Isotropy subgroup	Fixed point subspace	Dimension
(a)	$\mathbf{Z}_2 \dot{+} \mathbf{S}_3 = \langle \nu_1, s_2, s_3 \rangle$	$(0,0,0,0,0,0)$	0
(b)$^+$	$\mathbf{S}_3^+ = \langle s_2, s_3 \rangle$	$(X_1^+, X_1^+, X_1^+, 0, 0, 0)$	1
(b)$^-$	$\mathbf{S}_3^- = \langle s_2\nu_1, s_3\nu_1 \rangle$	$(0, 0, 0, X_1^-, -X_1^-, -X_1^-)$	1
(d)	$\mathbf{Z}_3 = \langle s_4 \rangle$	$(X_1^+, X_1^+, X_1^+, X_1^-, -X_1^-, -X_1^-)$	2
(g)$^+$	$\mathbf{S}_2^+ = \langle s_6 \rangle$	$(X_1^+, X_2^+, X_2^+, 0, X_2^-, -X_2^-)$	3
(g)$^-$	$\mathbf{S}_2^- = \langle s_6\nu_1 \rangle$	$(0, X_2^+, -X_2^+, X_1^-, X_2^-, X_2^-)$	3
(i)	$\mathbf{1}$	$(X_1^+, X_2^+, X_3^+, X_1^-, X_2^-, X_3^-)$	6

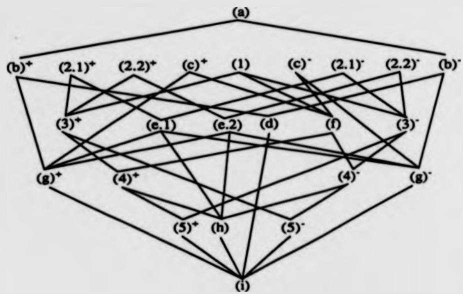
and the isotropy lattice is as follows:



In this case the fixed point subspaces that are missing by comparision with the table obtained for $H_3 = 0$ are (c)$^{\pm}$, (e), (f) and (h). Some of them will be recovered by considering all the symmetries that fix proper subspaces of Fix(K). Under the assumption $h_1, h_2, h_3 > 1$ the result is as follows:

Label	Isotropy subgroup	Fixed point subspace	Dimension
(c)$^+$	$((\mathbf{Z}^1_{k_2} \oplus \mathbf{Z}^2_{k_2}) \dot{+} \langle s_2 \rangle) \oplus \mathbf{Z}^3_{k_1}$	$(0, 0, X_3^+, 0, 0, 0)$	1
(c)$^-$	$((\mathbf{Z}^1_{k_2} \oplus \mathbf{Z}^2_{k_2}) \dot{+} \langle s_2\nu_1 \rangle) \oplus \mathbf{Z}^3_{k_1}$	$(0, 0, 0, 0, 0, X_3^-)$	1
(1)	$\mathbf{Z}^1_{k_1} \oplus \mathbf{Z}^2_{k_2} \oplus \mathbf{Z}^3_{k_1}$	$(0, 0, X_3^+, 0, 0, X_3^+)$	1
(2.1)$^+$	$\mathbf{Z}^1_{k_3} \oplus ((\mathbf{Z}^2_{k_1} \oplus \mathbf{Z}^3_{k_1}) \dot{+} \langle s_6 \rangle)$	$(0, X_2^+, X_2^+, 0, -X_2^+, X_2^+)$	1
(2.1)$^-$	$\mathbf{Z}^1_{k_2} \oplus ((\mathbf{Z}^2_{k_3} \oplus \mathbf{Z}^3_{k_3}) \dot{+} \langle s_6 \rangle)$	$(0, X_2^+, X_2^+, 0, X_2^+, -X_2^+)$	1
(2.2)$^+$	$\mathbf{Z}^1_{k_3} \oplus ((\mathbf{Z}^2_{k_1} \oplus \mathbf{Z}^3_{k_1}) \dot{+} \langle s_6\nu_1 \rangle)$	$(0, -X_2^+, X_2^+, 0, X_2^+, X_2^+)$	1
(2.2)$^-$	$\mathbf{Z}^1_{k_2} \oplus ((\mathbf{Z}^2_{k_3} \oplus \mathbf{Z}^3_{k_3}) \dot{+} \langle s_6\nu_1 \rangle)$	$(0, X_2^+, -X_2^+, 0, X_2^+, X_2^+)$	1
(e.1)	$\mathbf{Z}^1_{k_2} \oplus \langle s_6 \rangle$	$(0, X_2^+, -X_2^+, 0, X_2^-, X_2^-)$	2
(e.2)	$\mathbf{Z}^1_{k_2} \oplus \langle s_6\nu_1 \rangle$	$(0, X_2^+, X_2^+, 0, X_2^-, -X_2^-)$	2
(f)	$\mathbf{Z}^1_{k_2} \oplus \mathbf{Z}^2_{k_2} \oplus \mathbf{Z}^3_{k_1}$	$(0, 0, X_3^+, 0, 0, X_3^-)$	2
(3)$^+$	$\mathbf{Z}^1_{k_3} \oplus \mathbf{Z}^2_{k_1} \oplus \mathbf{Z}^3_{k_1}$	$(0, X_2^+, X_3^+, 0, -X_2^+, X_3^+)$	2
(3)$^-$	$\mathbf{Z}^1_{k_2} \oplus \mathbf{Z}^2_{k_3} \oplus \mathbf{Z}^3_{k_3}$	$(0, X_2^+, X_3^+, 0, X_2^+, -X_3^+)$	2
(4)$^+$	$\mathbf{Z}^1_{k_2} \oplus \mathbf{Z}^3_{k_1}$	$(0, X_2^+, X_3^+, 0, -X_2^+, X_3^-)$	3
(4)$^-$	$\mathbf{Z}^1_{k_3} \oplus \mathbf{Z}^3_{k_3}$	$(0, X_2^+, X_3^+, 0, X_2^+, X_3^-)$	3
(h)	$\mathbf{Z}^1_{k_2}$	$(0, X_2^+, X_3^+, 0, X_2^-, X_3^-)$	4
(5)$^+$	$\mathbf{Z}^3_{k_1}$	$(X_1^+, X_2^+, X_3^+, -X_1^+, -X_2^+, X_3^-)$	4
(5)$^-$	$\mathbf{Z}^3_{k_3}$	$(X_1^+, X_2^+, X_3^+, X_1^+, X_2^+, X_3^-)$	4

In this case the equivariant branching lemma guarantees the same branches as in the case k_1 odd, k_2, k_3 even. Note that here the subspaces (2.1)$^{\pm}$ and (2.2)$^{\pm}$ are no longer conjugate, the same happening with (e.1) and (e.2). By including these new subgroups in the isotropy lattice we get



3.6.4 $k_1 \neq k_2 \neq k_3, k_1 \neq k_3$: Invariant Theory

The invariants of the NBC problem are generated by the $\mathbf{T}^3$-invariants restricted to $\mathrm{Fix}(K)$ and symmetrized over the permutations $\mathbf{S}_3$. The restricted $\mathbf{T}^3$-invariants are generated by

$$x_1^2, \ldots, x_6^2 \qquad \text{and} \qquad \prod_{j=1}^{6} x_j^{a^j}$$

where

$$a^j = \max(|a_1^j|, |a_2^j|, |a_3^j|) \tag{3.20}$$

for $1 \leq j \leq 6$ and the a_i^j are such that

$$a_1^j \equiv a_2^j \equiv a_3^j \pmod 2 \tag{3.21}$$

and satisfy the equations

$$\begin{aligned} k_1(a_1^1 + a_1^6) + k_2(a_1^2 + a_1^4) + k_3(a_1^3 + a_1^5) &= 0 \\ k_1(a_2^2 + a_2^5) + k_2(a_2^1 + a_2^3) + k_3(a_2^4 + a_2^6) &= 0 \\ k_1(a_3^3 + a_3^4) + k_2(a_3^5 + a_3^6) + k_3(a_3^1 + a_3^2) &= 0. \end{aligned} \tag{3.22}$$

Each of these equations defines a codimension 1 lattice in $\mathbb{Z}^6$ and the only way we found to obtain a minimal set of generators is via an algorithm. This does not provide any general results because it restricts the solution to a case by case analysis. Instead we extract some general information about these lattices and use this to impose some constraints on the candidates to restricted invariants.

Lemma 7 *The following monomials are not restricted* $\mathbf{T}^3$*-invariants:*

1. *If* $h_2 > 1$
   $x_1 r$, $x_6 r$ *where* $r \in \langle x_2, x_3, x_4, x_5 \rangle$
   $x_2 r$, $x_5 r$ *where* $r \in \langle x_1, x_3, x_4, x_6 \rangle$
   $x_3 r$, $x_4 r$ *where* $r \in \langle x_1, x_2, x_5, x_6 \rangle$.

2. *If* $h_1 > 1$ *or* $h_3 > 1$
   $x_1 x_6 r$ *where* $r \in \langle x_2, x_5 \rangle$ *or* $r \in \langle x_3, x_4 \rangle$
   $x_2 x_5 r$ *where* $r \in \langle x_1, x_6 \rangle$ *or* $r \in \langle x_3, x_4 \rangle$
   $x_3 x_4 r$ *where* $r \in \langle x_1, x_6 \rangle$ *or* $r \in \langle x_2, x_5 \rangle$.

Proof Each item of this lemma will be proved individually.

1. We concentrate on the monomials $x_1 r$ and the others are analogous. A monomial of this form requires

   $$a^1 = 1 \qquad \text{and} \qquad a^6 = 0$$

   which is equivalent to

   $$a_j^1 = \pm 1 \qquad \text{and} \qquad a_j^6 = 0$$

for $1 \leq j \leq 3$. Recalling the assumption that k_1, k_2, k_3 are coprime we have from the first equation of (3.22) that

$$a_1^1 + a_1^6 \equiv 0 \pmod{h_2}.$$

Substituting the required values we get

$$\pm 1 \equiv 0 \pmod{h_2}$$

which contradicts the assumption $h_2 > 1$.

2. We prove for the monomials $x_1 x_6 r$ where $r \in \langle x_2, x_5 \rangle$ and the others are analogous. These monomials require

$$a_j^1, a_j^6 = \pm 1 \qquad \text{and} \qquad a_j^3, a_j^4 = 0$$

for $1 \leq j \leq 3$. From the second equation of (3.22) we have that

$$a_2^1 + a_2^3 \equiv 0 \pmod{h_3} \qquad \text{and} \qquad a_2^4 + a_2^6 \equiv 0 \pmod{h_1}.$$

Substituting the required values we get

$$\pm 1 \equiv 0 \pmod{h_3} \qquad \text{and} \qquad \pm 1 \equiv 0 \pmod{h_1}$$

which contradicts the assumption $h_1 > 1$ or $h_3 > 1$.

$\square$

We proceed by computing the H_3-invariants, in the coordinats X^+, X^- defined above, and eliminate the ones that we can show do not satisfy the equations (3.22).

Note that in part 1 of lemma 7 we are imposing a condition of $\mathbf{Z}_{h_2}$ invariance and part 2 has an assumption of $\mathbf{Z}_{h_1}$ and $\mathbf{Z}_{h_3}$ invariance.

The normalizer quotient H_3 is isomorphic to $T \dot{+} \mathbf{S}_3$ where T consists of the elements of $\mathbf{T}^3$ that leave $\mathrm{Fix}(K)$ invariant. Recall that T is isomorphic to $(\mathbf{Z}_2)^3$, $(\mathbf{Z}_2)^2$ or $\mathbf{Z}_2$ depending on the parity of the mode numbers. We begin by computing the generators of the T-invariants and then symmetrize these over the permutations $\mathbf{S}_3$.

Lemma 8 *The T-invariants are generated by the monomials as follows:*

(a) *If k_1 odd, k_2, k_3 even*

$$U_1^+, \quad U_2^+, \quad U_3^+, \quad U_1^-, \quad U_2^-, \quad U_3^-, \quad X_1^+X_1^-, \quad X_2^+X_2^-, \quad X_3^+X_3^-$$

(b) *If k_1 even, k_2, k_3 odd*

$$\begin{gathered} U_1^+, \quad U_2^+, \quad U_3^+, \quad U_1^-, \quad U_2^-, \quad U_3^-, \quad X_1^+X_1^-, \quad X_2^+X_2^-, \quad X_3^+X_3^-, \\ X_1^+X_2^+X_3^+, \quad X_1^+X_2^-X_3^-, \quad X_1^+X_2^-X_3^+, \quad X_1^-X_2^+X_3^-, \quad X_1^+X_2^+X_3^- \end{gathered}$$

(c) *If k_1, k_2, k_3 odd*

$$\begin{gathered} U_1^+, \quad U_2^+, \quad U_3^+, \quad U_1^-, \quad U_2^-, \quad U_3^-, \quad X_1^+X_1^-, \quad X_2^+X_2^-, \quad X_3^+X_3^-, \\ X_1^+X_2^+, \quad X_2^+X_3^+, \quad X_1^+X_3^+, \quad X_1^-X_2^-, \quad X_2^-X_3^-, \quad X_1^-X_3^-, \\ X_1^+X_2^-, \quad X_2^+X_3^-, \quad X_1^+X_3^-, \quad X_1^-X_2^+, \quad X_2^-X_3^+, \quad X_1^-X_3^+ \end{gathered}$$

where $U_j^{\pm} = (X_j^{\pm})^2$ for $1 \leq j \leq 3$.

Proof In case (a),(b),(c) the group T is isomorphic to $(\mathbf{Z}_2)^3$, $(\mathbf{Z}_2)^2$, $\mathbf{Z}_2$ respectively. This group is generated by ν_1, ν_2, ν_3 as in section 3.6.4. The result comes by applying these reflections to a generic monomial. □

Proposition 9 *The H_3-invariants are generated up to fourth order as follows:*

(a) *If k_1 odd, k_2, k_3 even*

$$
\begin{aligned}
I_1 &= U_1^+ + U_2^+ + U_3^+ \\
I_2 &= U_1^- + U_2^- + U_3^- \\
I_3 &= U_1^+U_2^+ + U_2^+U_3^+ + U_1^+U_3^+ \\
I_4 &= U_1^-U_2^- + U_2^-U_3^- + U_1^-U_3^- \\
I_5 &= U_1^+U_2^- + U_2^+U_1^- + U_3^+U_2^- + U_3^+U_1^- + U_2^+U_3^- + U_1^+U_3^- \\
I_6 &= X_1^+X_1^-X_2^+X_2^- - X_2^+X_2^-X_3^+X_3^- + X_1^+X_1^-X_3^+X_3^- \\
I_7 &= X_1^+X_1^-(U_2^+ - X_3^+) + X_2^+X_2^-(U_1^+ - X_3^+) + X_3^+X_3^-(U_2^+ - X_1^+) \\
I_8 &= X_1^+X_1^-(U_2^- - X_3^-) + X_2^+X_2^-(U_1^- - X_3^-) + X_3^+X_3^-(U_2^- - X_1^-).
\end{aligned}
$$

(b) *If k_1 even, k_2, k_3 odd*

$$
\begin{aligned}
I_j & \quad \textit{for } 1 \leq j \leq 8 \textit{ as in (a)} \\
I_9 &= X_1^+X_2^+X_3^+ \\
I_{10} &= X_1^-X_2^-X_3^+ - X_2^-X_3^-X_1^+ + X_1^-X_3^-X_2^+.
\end{aligned}
$$

(c) *If k_1, k_2, k_3 odd*

$$
\begin{aligned}
I_j & \quad \textit{for } 1 \leq j \leq 8 \textit{ as in (a)} \\
I_9 &= X_1^+X_2^+ + X_2^+X_3^+ + X_1^+X_3^+ \\
I_{10} &= X_1^-X_2^- - X_2^-X_3^- + X_1^-X_3^- \\
I_{11} &= X_1^+X_2^- + X_2^+X_1^- - X_3^+X_2^- - X_3^+X_1^- + X_2^+X_3^- - X_1^+X_3^- \\
I_{12} &= U_1^+X_2^-X_3^- - U_2^+X_1^-X_3^- - U_3^+X_1^-X_2^- \\
I_{13} &= U_1^-X_2^+X_3^+ + U_2^-X_1^+X_3^+ + U_3^-X_1^+X_2^+ \\
I_{14} &= U_1^+X_1^+(X_2^- - X_3^-) + U_2^+X_2^+(X_1^- + X_3^-) - U_3^+X_3^+(X_2^- + X_1^-) \\
I_{15} &= U_1^+X_1^-(X_2^+ - X_3^+) + U_2^+X_2^-(X_1^+ - X_3^+) + U_3^+X_3^-(X_2^+ - X_1^+) \\
I_{16} &= U_1^-X_1^-(X_2^+ - X_3^+) + U_2^-X_2^-(X_1^+ - X_3^+) + U_3^-X_3^-(X_2^+ - X_1^+) \\
I_{17} &= U_1^-X_1^+(X_2^- - X_3^-) + U_2^-X_2^+(X_1^- + X_3^-) - U_3^-X_3^+(X_2^- + X_1^-) \\
I_{18} &= U_1^+(X_2^+X_2^- - X_3^+X_3^-) + U_2^+(X_1^+X_1^- + X_3^+X_3^-) - U_3^+(X_2^+X_2^- + X_1^+X_1^-) \\
I_{19} &= U_1^-(X_2^+X_2^- - X_3^+X_3^-) + U_2^-(X_1^+X_1^- + X_3^+X_3^-) - U_3^-(X_2^+X_2^- + X_1^+X_1^-).
\end{aligned}
$$

Proof Straightforward calculation using the monomials in lemma 8. □

As a final step to get our best approach of the restricted invariants we eliminate from proposition 9 the ones that we know do not satisfy the equations (3.22).

Proposition 10 *The restricted* $O\dot{+}S_3$*-invariants up to fourth order are contained in the ring generated as follows:*

(a) *If* k_1 *odd,* k_2, k_3 *even*

- I_j *for* $1 \le j \le 8$ *if* $h_1 = h_3 = 1$
- $I_1 + I_2$, $I_3 + I_4 + I_5$, I_6, $I_7 + I_8$ *otherwise.*

(b) *If* k_1 *even,* k_2, k_3 *odd*

- *Symmetrization of* I_j *for* $1 \le j \le 8$ *depending on* h_1, h_3 *as in (a)*
- I_9, I_{10} *if* $h_2 = 1$.

(c) *If* k_1, k_2, k_3 *odd*

- *Symmetrization of* I_j *for* $1 \le j \le 8$ *depending on* h_1, h_3 *as in (a)*
- I_j *for* $9 \le j \le 19$ *if* $h_2 = 1$.

Proof

(a) By writing I_1, I_2 in the initial coordinates we get

$$\begin{aligned} I_1 &= (x_1 + x_6)^2 + (x_2 + x_5)^2 + (x_3 + x_4)^2 \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2(x_1^2x_6^2 + x_2^2x_5^2 + x_3^2x_4^2) \\ I_2 &= (x_1 - x_6)^2 + (x_2 - x_5)^2 + (x_3 - x_4)^2 \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 - 2(x_1^2x_6^2 + x_2^2x_5^2 + x_3^2x_4^2). \end{aligned}$$

Now $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ is a restricted $O\dot{+}S_3$-invariant and by part 2 of lemma 7, if $h_1 > 1$ or $h_3 > 1$ the polynomial $x_1^2x_6^2 + x_2^2x_5^2 + x_3^2x_4^2$ cannot be extended to a $\mathbf{T}^3$-invariant. This part is eliminated by adding I_1 and I_2

$$I_1 + I_2 = (U_1^+ + U_1^-) + (U_2^+ + U_2^-) + (U_3^+ + U_3^-).$$

Now we write $U_1^+U_2^+$, $U_1^-U_2^-$ and $U_1^+U_2^- + U_2^+U_1^-$ in the initial coordinates as follows:

$$\begin{aligned} U_1^+U_2^+ &= (x_1 + x_6)^2(x_2 + x_5)^2 \\ &= x_1^2x_2^2 + x_1^2x_5^2 + x_2^2x_6^2 + x_5^2x_6^2 + 4x_1x_2x_5x_6 \\ &\quad +2(x_1^2x_2x_5 + x_6^2x_2x_5 + x_2^2x_1x_6 + x_5^2x_1x_6) \\ U_1^-U_2^- &= (x_1 - x_6)^2(x_2 - x_5)^2 \\ &= x_1^2x_2^2 + x_1^2x_5^2 + x_2^2x_6^2 + x_5^2x_6^2 + 4x_1x_2x_5x_6 \\ &\quad -2(x_1^2x_2x_5 + x_6^2x_2x_5 + x_2^2x_1x_6 + x_5^2x_1x_6) \\ U_1^+U_2^- + U_2^+U_1^- &= (x_1 + x_6)^2(x_2 - x_5)^2 + (x_2 + x_5)^2(x_1 - x_6)^2 \\ &= 2(x_1^2x_2^2 + x_1^2x_5^2 + x_2^2x_6^2 + x_5^2x_6^2) - 8x_1x_2x_5x_6 \end{aligned}$$

The polynomial $x_1^2x_2^2+x_1^2x_5^2+x_2^2x_6^2+x_5^2x_6^2$ is a restricted $\mathbf{T}^3$-invariant by and part 2 of lemma 3.22, if $h_1 > 1$ or $h_3 > 1$ the monomial $x_1x_2x_5x_6$ cannot be extended to a $\mathbf{T}^3$-invariant. This is eliminated by the addition

$$U_1^+U_2^+ + U_1^-U_2^- + U_1^+U_2^- + U_2^+U_1^- \quad = \quad (U_1^+ + U_1^-)(U_2^+ + U_2^-).$$

By analogy the same holds for $U_2^\pm U_3^\pm$ and $U_1^\pm U_3^\pm$. Therefore the symmetrization of I_1, I_2, I_3 if $h_1 > 1$ or $k_3 > 1$ is

$$\begin{aligned} I_3 + I_4 + I_5 \quad = \quad & (U_1^+ + U_1^-)(U_2^+ + U_2^-) + (U_2^+ + U_2^-)(U_3^+ + U_3^-) \\ & +(U_1^+ + U_1^-)(U_3^+ + U_3^-). \end{aligned}$$

(b) Comes from part 1 of lemma 7 by recalling that $X^\pm$ are obtained by a linear change of the original coordinates.

(c) Same as part (b).

□

By playing a bit with combinatorics we see that if $h_2 > 1$ the restricted equivariants are generated up to third order as follows:

- If $h_1 = h_3 = 1$

$$E_1^+ = \begin{pmatrix} X_1^+ \\ X_2^+ \\ X_3^+ \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_1^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^- \\ X_2^- \\ X_3^- \end{pmatrix}$$

$$E_2^+ = \begin{pmatrix} X_1^+U_1^+ \\ X_2^+U_2^+ \\ X_3^+U_3^+ \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_2^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^-U_1^- \\ X_2^-U_2^- \\ X_3^-U_3^- \end{pmatrix}$$

$$E_3^+ = \begin{pmatrix} X_1^+U_1^- \\ X_2^+U_2^- \\ X_3^+U_3^- \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_3^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^-U_1^+ \\ X_2^-U_2^+ \\ X_3^-U_3^+ \end{pmatrix}$$

$$E_4^+ = \begin{pmatrix} X_1^-(X_2^+X_2^- + X_3^+X_3^-) \\ X_2^-(X_1^+X_1^- - X_3^+X_3^-) \\ X_3^-(X_1^+X_1^- - X_2^+X_2^-) \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_4^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^+(X_2^+X_2^- + X_3^+X_3^-) \\ X_2^+(X_1^+X_1^- - X_3^+X_3^-) \\ X_3^+(X_1^+X_1^- - X_2^+X_2^-) \end{pmatrix}$$

$$E_5^+ = \begin{pmatrix} X_1^+(X_2^+X_2^- - X_3^+X_3^-) \\ X_2^+(X_1^+X_1^- + X_3^+X_3^-) \\ -X_3^+(X_1^+X_1^- + X_2^+X_2^-) \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_5^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^-(X_2^+X_2^- - X_3^+X_3^-) \\ X_2^-(X_1^+X_1^- + X_3^+X_3^-) \\ -X_3^-(X_1^+X_1^- + X_2^+X_2^-) \end{pmatrix}$$

$$E_6^+ = \begin{pmatrix} X_1^-(U_2^+ - U_3^+) \\ X_2^-(U_1^+ - U_3^+) \\ X_3^-(U_2^+ - U_1^+) \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_6^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^+(U_2^- - U_3^-) \\ X_2^+(U_1^- - U_3^-) \\ X_3^+(U_2^- - U_1^-) \end{pmatrix}$$

$$E_7^+ = \begin{pmatrix} X_1^-(U_2^- - U_3^-) \\ X_2^-(U_1^- - U_3^-) \\ X_3^-(U_2^- - U_1^-) \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E_7^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_1^+(U_2^+ - U_3^+) \\ X_2^+(U_1^+ - U_3^+) \\ X_3^+(U_2^+ - U_1^+) \end{pmatrix}.$$

- Otherwise

$$E_1^+ + E_1^-, \quad E_2^+ + E_2^-, \quad E_3^+ + E_3^-, \quad E_4^+ + E_4^-, \quad E_5^+ + E_5^-, \quad E_6^+ + E_6^-, \quad E_7^+ + E_7^-.$$

Theorem 16 *Assume that $\mathcal{P}(u) = 0$ satisfying NBC on the cube $[0,\pi] \times [0,\pi] \times [0,\pi]$ undergoes a single mode bifurcation with mode numbers $(k_1, k_2, k_3) \in \mathbb{N}^3$ when the parameter λ crosses zero. Then, if $h_2 > 1$, the third order truncation of the bifurcation equations on $\ker L$ is*

$$\begin{aligned}
f_1^+(X,\lambda) &= a^+X_1^+ + b^+X_1^+U_1^+ + c^+X_1^+U_1^- + d^+X_1^-(X_2^+X_2^- + X_3^+X_3^-) + \\
&\quad + e^+X_1^+(X_2^+X_2^- - X_3^+X_3^-) + p^+X_1^-(U_2^+ - U_3^+) + q^+X_1^-(U_2^- - U_3^-) = 0 \\[2ex]
f_2^+(X,\lambda) &= a^+X_2^+ + b^+X_2^+U_2^+ + c^+X_2^+U_2^- + d^+X_2^-(X_1^+X_1^- - X_3^+X_3^-) + \\
&\quad + e^+X_2^+(X_1^+X_1^- + X_3^+X_3^-) + p^+X_2^-(U_1^+ - U_3^+) + q^+X_2^-(U_1^- - U_3^-) = 0 \\[2ex]
f_3^+(X,\lambda) &= a^+X_3^+ + b^+X_3^+U_3^+ + c^+X_3^+U_3^- + d^+X_3^-(X_1^+X_1^- - X_2^+X_2^-) + \\
&\quad + e^+X_3^+(X_1^+X_1^- + X_2^+X_2^-) + p^+X_3^-(U_2^+ - U_1^+) + q^+X_3^-(U_2^- - U_1^-) = 0 \\[2ex]
f_1^-(X,\lambda) &= a^-X_1^- + b^-X_1^-U_1^- + c^-X_1^-U_1^+ + d^-X_1^+(X_2^+X_2^- + X_3^+X_3^-) + \\
&\quad + e^-X_1^-(X_2^+X_2^- - X_3^+X_3^-) + p^-X_1^+(U_2^- - U_3^-) + q^-X_1^+(U_2^+ - U_3^+) = 0 \\[2ex]
f_2^-(X,\lambda) &= a^-X_2^- + b^-X_2^-U_2^- + c^-X_2^-U_2^+ + d^-X_2^+(X_1^+X_1^- - X_3^+X_3^-) + \\
&\quad + e^-X_2^-(X_1^+X_1^- + X_3^+X_3^-) + p^-X_2^+(U_1^- - U_3^-) + q^-X_2^+(U_1^+ - U_3^+) = 0 \\[2ex]
f_3^-(X,\lambda) &= a^-X_3^- + b^-X_3^-U_3^- + c^-X_3^-U_3^+ + d^-X_3^+(X_1^+X_1^- - X_2^+X_2^-) + \\
&\quad - e^-X_3^-(X_1^+X_1^- + X_2^+X_2^-) + p^-X_3^+(U_2^- - U_1^-) + q^-X_3^+(U_2^+ - U_1^+) = 0
\end{aligned}$$

where

- *$a^{\pm}$ are linear functions of I_1, I_2 and λ, and $b^{\pm}, c^{\pm}, d^{\pm}, e^{\pm}, p^{\pm}, q^{\pm}$ are linear functions of λ if $h_1 = h_3 = 1$;*

- *$a^+ = a^-$ are linear functions of $I_1 + I_2$ and λ, and $b^+ = b^- = c^+ = c^-$, $d^+ = d^- = e^+ = e^-$, $p^+ = p^- = q^+ = q^-$ are linear functions of λ otherwise.*

Proof Denote

$$\begin{aligned} I_3' &= (U_1^+)^2 + (U_2^+)^2 + (U_3^+)^2 \\ I_4' &= (U_1^-)^2 + (U_2^-)^2 + (U_3^-)^2 \\ I_5' &= 2(U_1^+U_1^- + U_2^+U_2^- + U_3^+U_3^-) \end{aligned}$$

It can be checked that

$$\begin{aligned} I_1^2 &= I_3' + 2I_3 \\ I_2^2 &= I_4' + 2I_4 \\ 2I_1I_2 &= I_5' + 2I_5. \end{aligned}$$

Therefore if we substitute I_3, I_4, I_5 by I_3', I_4', I_5' in proposition 9. the result is the same. Proposition 10 can also be adapted and say that $I_1, I_2, I_3', I_4', I_5'$ generate the restricted $\mathbf{O}\dot{+}\mathbf{T}^3$-invariants if $h_2 > 1$ and $h_1 = h_3 = 1$. Then an equivariant is of the form

$$a^+\nabla I_1 + a^-\nabla I_2 + b^+\nabla I_3' + b^-\nabla I_4' + c\nabla I_5' + \text{hot}$$

where $a^{\pm}, b^{\pm}, c$ are functions of the invariants I_1, I_2, I_3, I_4, I_5. By including the bifurcation parameter λ and truncating at cubic order we get the first item of the theorem.

Now by proposition 10, if $h_2 > 1$ and at least one of $h_1, h_3 > 1$ the restricted $\mathbf{O}\dot{+}\mathbf{T}^3$-invariants are generated by $I_1 + I_2$, $I_3 + I_4 + I_5$. It can be checked that

$$(I_1^2 + I_2^2)^2 = (I_3' + I_4' + I_5') + 2(I_3 + I_4 + I_5).$$

Thus, the result in proposition 10 is the same if $I_3 + I_4 + I_5$ is substituted by $I_3' + I_4' + I_5'$. Then an equivariant is of the form

$$a\nabla(I_1 + I_2) + b\nabla(I_3' + I_4' + I_5') + \text{hot}$$

where a, b are functions of the invariants $I_1 + I_2$, $I_3 + I_4 + I_5$. By including the bifurcation parameter λ and truncating at cubic order we get the second item of the theorem. □

We did some calculations with the aim of obtaining the branching equations and stability for the large class of bifurcation equations satisfying the conditions $h_2 > 1$ and at least one of $h_1, h_3 = 1$. Under this assumption, by theorem 16, the truncation of the bifurcation equations at cubic order is independent of the exact value of the mode numbers. We saw that the low order terms are enough to give the criticality of the branches as well as their stability. The results are not shown here because they are too extensive and tedious. We end this section with two remarks.

Remarks Here we make two remarks on the constraints imposed on the highest common factors h_1, h_2, h_3.

1. The condition $h_2 > 1$ holds automatically if k_1 odd, k_2, k_3 even but not necessarily otherwise. Without this condition the bifurcation equations have less symmetry. We need more equivariants, some of which may have an important effect in the bifurcation diagrams (such as reduce the number of branches). It seems reasonably easy to explore this situation if k_1 even, k_2, k_3 odd but it is much more complicated if all the mode numbers are odd. This problem is left open for the time being.

2. If both $h_1, h_3 > 1$ the bifurcation equations have more symmetry. It can be easily checked that in this case the cubic truncation is not enough to give the direction of branching and stability. The order of the terms needed depends strongly on h_1, h_3. A classification of bifurcation diagrams could be done for small h_1, h_3 but we may not do it here.

Chapter 4

Three Dimensional Bénard Convection

4.1 Introduction

In this chapter we apply some of the results of chapters 2 and 3 to the Bénard convection problem in a 3-dimensional box. In section 1.1 we gave a brief description of bifurcations occurring in a 2-dimensional vertical cross section of the domain. By exploring the third direction we find a much richer structure of patterns, specially when the horizontal cross section is a square.

In section 4.2 we state the Boussinesq approximation of the equations for time independent convection in a box with a mixture of Neumann and Dirichlet boundary conditions. Then we define a scaling that makes the quantities nondimensional.

In section 4.3 we set the symmetry context where these equations should be viewed. It will be shown that the problem has more symmetries than the group that leaves the domain invariant. These symmetries are found by defining a periodic extension by reflection across the boundaries in such a way that the regularity of the solutions is preserved. The extended problem satisfies periodic boundary conditions on a larger domain and the symmetries are a 3-torus extension of the group that leaves the domain invariant. Then, the solutions that satisfy the original boundary conditions are constrained by invariance under a group of reflections.

Sections 4.4 and 4.5 deal with bifurcation problems. The construction described above will be used to give a general form of the reduced bifurcation equations. In appendices A and B, normal forms and universal unfoldings are given together with some bifurcation diagrams.

Sections 4.6, 4.7 and 4.8 are concerned with the well known method of Liapunov-Schmidt reduction that gives an exact Taylor expansion of the bifurcation equations. From the previous sections we have the information about which terms must be calculated. By taking the result to appendix A or B we get the bifurcation diagrams.

4.2 The Boussinesq Equations

As in Golubitsky *et al.* [11], the Boussinesq approximation of the equations for the Bénard convection in the box $[0, \pi\ell_1] \times [0, \pi\ell_2] \times [0, \pi\ell_3]$ may be written as

$$\begin{aligned}
&\frac{1}{\sigma}\left(\frac{\partial v}{\partial t} + (v.\nabla)v\right) = -\nabla p + \Theta g + \Delta v \\
&\operatorname{div} v = 0 \\
&\frac{\partial \Theta}{\partial t} + (v.\nabla)\Theta = R v_3 + \Delta\Theta
\end{aligned}$$

where $v = (v_1, v_2, v_3)$ is the velocity vector, Θ describes the deviation of temperature, p is the pressure, g is gravity and the parameters R and σ are, respectively, the Rayleigh number and the Prandtl number. We scale the domain variables as

$$\xi_1 \mapsto \frac{1}{\ell_1}\xi_1 \qquad \xi_2 \mapsto \frac{1}{\ell_2}\xi_2 \qquad \xi_3 \mapsto \frac{1}{\ell_3}\xi_3,$$

the deviation of temperature, the pressure and the Rayleigh and Prandtl numbers are scaled as

$$\Theta \mapsto \ell_3^3\Theta \qquad p \mapsto \ell_3 p \qquad R \mapsto \ell_3^2 R \qquad \sigma \mapsto \frac{1}{\ell_3}\sigma.$$

By denoting $u = (v, \Theta, p)$ and applying the scaling above to the time independent Boussinesq equations we get

$$\begin{aligned}
\Phi_r[1](u,R) &\equiv \Delta_r v_1 - \frac{1}{r_1}\frac{\partial p}{\partial \xi_1} - \frac{1}{\sigma} v.\nabla_r v_1 = 0 \\
\Phi_r[2](u,R) &\equiv \Delta_r v_2 - \frac{1}{r_2}\frac{\partial p}{\partial \xi_2} - \frac{1}{\sigma} v.\nabla_r v_2 = 0 \\
\Phi_r[3](u,R) &\equiv \Delta_r v_3 - \frac{\partial p}{\partial \xi_3} + \Theta - \frac{1}{\sigma} v.\nabla_r v_3 = 0 \\
\Phi_r[4](u,R) &\equiv \Delta_r \Theta + R v_3 - v.\nabla_r \Theta = 0 \\
\Phi_r[5](u,R) &\equiv \nabla_r . v = 0
\end{aligned} \tag{4.1}$$

where the parameter

$$r = (r_1, r_2) = \left(\frac{\ell_1}{\ell_3}, \frac{\ell_2}{\ell_3}\right)$$

has been transferred from the dimensions of the domain to the equations, and Δ_r, ∇_r are, respectively, scaled Laplacian and gradient as

$$\begin{aligned}
\Delta_r &= \frac{1}{r_1^2}\frac{\partial^2}{\partial \xi_1^2} + \frac{1}{r_2^2}\frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} \\
\nabla_r &= \left(\frac{1}{r_1}\frac{\partial}{\partial \xi_1}, \frac{1}{r_2}\frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}\right).
\end{aligned}$$

We denote the scaled domain $\Omega = [0,\pi]^3$ and the boundary conditions are a mixture of Neumann and Dirichlet as

$$\begin{aligned} v_1 = \frac{\partial v_2}{\partial \xi_1} = \frac{\partial v_3}{\partial \xi_1} = \Theta_{\xi_1} = 0 \qquad &\text{for} \qquad \xi_1 = 0,\pi \\ \frac{\partial v_1}{\partial \xi_2} = v_2 = \frac{\partial v_3}{\partial \xi_2} = \Theta_{\xi_2} = 0 \qquad &\text{for} \qquad \xi_2 = 0,\pi \\ \frac{\partial v_1}{\partial \xi_3} = \frac{\partial v_2}{\partial \xi_3} = v_3 = \Theta = 0 \qquad &\text{for} \qquad \xi_3 = 0,\pi. \end{aligned} \tag{4.2}$$

4.3 The Appropriate Symmetry Context

Let $u \in (C^\infty(\Omega))^5$ be an arbitrary solution of $\Phi_r = 0$. We proceed by summarizing the extension method described in chapters 2 and 3 and showing that the procedure applies to this particular problem. We define an extension $\hat{u}$ by reflecting each component of u across the boundaries of the domain Ω and combine it with sign change in the case of a Dirichlet boundary condition. These reflections generate the group

$$K = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \langle \kappa_1, \kappa_2, \kappa_3 \rangle$$

acting on $\hat{u}$ as

$$\begin{aligned} \kappa_1 &: \ (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (-v_1,v_2,v_3,\Theta,p)(-\xi_1,\xi_2,\xi_3) \\ \kappa_2 &: \ (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (v_1,-v_2,v_3,\Theta,p)(\xi_1,-\xi_2,\xi_3) \\ \kappa_3 &: \ (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (v_1,v_2,-v_3,-\Theta,p)(\xi_1,\xi_2,-\xi_3). \end{aligned}$$

In lemma 9 below we show that this extended function is a solution of the same operator on the larger domain

$$\hat{\Omega} = [-\pi,\pi]^3.$$

Then, by extending $\hat{u}$ periodically to $\mathbf{R}^3$ we still have a solution of $\Phi_r = 0$ for which we keep the notation $\hat{u}$. Lemma 10 below says that this extension preserves the regularity of the solution. The extended solution $\hat{u}$ satisfies periodic boundary conditions on $\hat{\Omega}$.

Now we define an action of the extended group $K \dot{+} \mathbf{T}^3$. This group is generated by the reflections $\kappa_1, \kappa_2, \kappa_3$ acting as above together with the translations $\theta_j \in [0, 2\pi)$ for $j = 1,2,3$ acting as

$$\begin{aligned} \theta_1 &: \ \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1+\theta_1,\xi_2,\xi_3) \\ \theta_2 &: \ \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1,\xi_2+\theta_2,\xi_3) \\ \theta_3 &: \ \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1,\xi_2,\xi_3+\theta_3). \end{aligned}$$

Given the translation invariance of the operator Φ_r, by acting on $\hat{u}$ with an element of the 3-torus $\mathbf{T}^3$ we get a solution of the periodic boundary value problem.

Up to this point we know that any smooth solution u of $\Phi_r = 0$ satisfying the boundary conditions (4.2) corresponds to a unique $K \dot{+} \mathbf{T}^3$-orbit of solutions to the same equation satisfying periodic boundary conditions on $\hat{\Omega}$. We choose from this group orbit of $\hat{u}$ all the solutions w that satisfy the boundary conditions (4.2). These are exactly the ones that are fixed by the action of K

$$\kappa w(\kappa\xi) = w(\xi)$$

for all $\kappa \in K$. These solutions belong to the subspace fixed by K, which we denote by $\mathrm{Fix}(K)$.

We proceed by stating and proving the results referred to above.

Lemma 9 *Let u be a solution of $\Phi_r = 0$. Let the group K act as above. Then κu is a solution of the same equation for all $\kappa \in K$.*

Proof Given the group structure of K it is enough to show that this result holds for the generators of K. We define an action of $\kappa_1, \kappa_2, \kappa_3$ on the operator Φ_r as

$$\begin{aligned}
\kappa_1 &: (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (-\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \\
\kappa_2 &: (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[1], -\Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \\
\kappa_3 &: (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[1], \Phi_r[2], -\Phi_r[3], -\Phi_r[4], \Phi_r[5]).
\end{aligned}$$

It is easy to see that Φ_r commutes with the action of K

$$\Phi_r(\kappa_j u(\kappa_j\xi), R) = \kappa_j \Phi_r(u(\xi), R) \qquad \text{for} \qquad j = 1, 2, 3. \tag{4.3}$$

By assumption, u is a solution of $\Phi_r = 0$. So

$$\Phi_r(u(\xi), R) = 0 \qquad \text{for} \qquad j = 1, 2, 3.$$

Together with (4.3) this implies that

$$\Phi_r(\kappa_j u(\kappa_j\xi), R) = 0 \qquad \text{for} \qquad j = 1, 2, 3$$

and this is what we wanted to show. □

Lemma 10 *Let $u \in (C^1(\mathbb{R}^3))^5$ be a solution of $\Phi_r = 0$. Then $u \in (C^\infty(\mathbb{R}^3))^5$.*

Proof See Field *et al.* [8]. □

Note that by this method we find symmetries that are not obvious in bounded domains. The most immediate thing to do would be to consider only the reflections that leave the equations and domain invariant. This approach would be simpler but incomplete: some translations of our extended solutions satisfy the required boundary conditions and remain hidden if we insist upon leaving the domain invariant.

Up to now we mentioned all the symmetries that do not depend on the parameter r. If we allow group actions on r there is one more symmetry: the group $\mathbf{S}_2$. This group has one generator denoted by s and acting as

$$\begin{aligned}
s &: (r_1, r_2) \mapsto (r_2, r_1) \\
s &: (v_1, v_2, v_3, \Theta, p)(\xi_1, \xi_2, \xi_3) \mapsto (v_2, v_1, v_3, \Theta, p)(\xi_2, \xi_1, \xi_3) \\
s &: (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[2], \Phi_r[1], \Phi_r[3], \Phi_r[4], \Phi_r[5]).
\end{aligned}$$

By noting that the set of boundary conditions is invariant under the action of $\mathbf{S}_2$ on u we have the following

Lemma 11 *Let u be a solution of $\Phi_r = 0$ satisfying the boundary conditions (4.2). Let the group $\mathbf{S}_2$ act as above. Then su is a zero of $\Phi_{sr} = 0$ satisfying the same boundary conditions.*

Proof As we said before there is nothing to prove about the boundary conditions. So we only have to show that the s-conjugate of a solution u of $\Phi_r = 0$ is a solution of the conjugate operator Φ_{sr}. A straightforward calculation shows that the operator Φ_r commutes with the action of $\mathbf{S}_2$ as

$$\Phi_{sr}(su(s\xi), R) \;=\; s\Phi_r(u(\xi), R). \tag{4.4}$$

By assumption, u is a solution of $\Phi_r = 0$. So

$$\Phi_r(u(\xi), R) \;=\; 0.$$

Together with (4.4) this implies that

$$\Phi_{sr}(su(s\xi), R) \;=\; 0$$

and the result follows. □

Note that if $r_1 = r_2$ the operator Φ_r is invariant under the action of $\mathbf{S}_2$. In this case, given a solution u of $\Phi_r = 0$ there is a conjugate su satisfying the same equations. If $r_1 \neq r_2$ then su is a solution of the different equation $\Phi_{sr} = 0$.

Finally we combine all the results obtained up to now in the following

Theorem 17 *Let Φ_r be the operator defined by (4.1). Let the groups K, $\mathbf{T}^3$ and $\mathbf{S}_2$ act as above. Then*

1. *If $r_1 \neq r_2$ ($r_1 = r_2$) every smooth K-invariant solution $\hat{u}$ of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K(\dot{+}\mathbf{S}_2)\dot{+}\mathbf{T}^3$-symmetry restricts to a smooth solution of $\Phi_r = 0$ on Ω with the boundary conditions (4.2).*

2. *Let $u \in (C^1(\Omega))^5$ be a solution of $\Phi_r = 0$ on Ω with the boundary conditions (4.2). Then*

    - *u is smooth.*
    - *If $r_1 \neq r_2$ then u extends uniquely to a smooth K-invariant solution of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K\dot{+}\mathbf{T}^3$-symmetry. The $\mathbf{S}_2$-conjugate su is a solution of the equation defined by the $\mathbf{S}_2$-conjugate operator Φ_{sr}.*
    - *If $r_1 = r_2$ then u extends uniquely to a smooth K-invariant solution of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-symmetry.*

We observe that $u = 0$ is always a translation invariant solution of the equation $\Phi_r = 0$. We are interested in steady states bifurcating from this trivial branch when the Rayleigh number R is increased from below. We restrict our bifurcation analysis to a neighbourhood of some critical values of the unfolding parameter r. Denote $L_r = d\Phi_r$ the linearization of Φ_r about $u = 0$. In sections 4.4 and 4.5 below, and according to chapters 2 and 3, we use the symmetries in theorem 17 to get a general polynomial form for the projection of the bifurcation equations onto ker L_r. In later sections a Liapunov-Schmidt reduction will be performed to give the exact values of the coefficients in the bifurcation equations. Finally, these equations will be put into normal form and the bifurcation diagrams read from the tables in appendix.

4.4 Bifurcations with a $K\dot{+}\mathbf{T}^3$-Symmetric Extension

Assume that r is such that when the parameter R is increased from zero, it crosses a critical value for which the linear operator L_r has a nontrivial kernel. Then there are branches of solutions bifurcating from $u = 0$. The Rayleigh number R is playing the role of bifurcation parameter and r is a 2-dimensional unfolding parameter. Recall that $r = (r_1, r_2)$ depends only on the domain before being scaled: the two components represent the aspect ratios of the horizontal dimensions of the box by the vertical dimension.

By Golubitsky *et al.* [11], bifurcations of codimension up to three are generic in problems with two unfolding parameters. Such bifurcations are expected to occur in regions of the unfolding parameter space as follows:

- Codimension one in open regions.
- Codimension two along lines.
- Codimension three at isolated points.

We begin by describing shortly the simplest codimension one bifurcations, proceeding then to the more complicated ones. The main interest of this section is the codimension three bifurcation.

As in section 4.3, from the problem $\Phi_r = 0$ with the boundary conditions (4.2) we construct a larger one consisting of the same equation with periodic boundary conditions. The second problem is invariant under an action of the group $K\dot{+}\mathbf{T}^3$. We also know that in order to get explicitly all the symmetries of the first, we really need to state the extended problem and restrict the result to the subspace fixed by K. This is what we proceed to do.

4.4.1 Single Mode Bifurcations

By Golubitsky *et al.* [11], we have a codimension one bifurcation with $K\dot{+}\mathbf{T}^3$-symmetry when ker L_r is an irreducible representation of this group. Suppose that r is such that this holds for some value of the bifurcation parameter R. We may write an irreducible representation of $\mathbf{T}^3$ as

$$\begin{aligned}
\theta_1 &: \ (z_1, z_2, z_3, z_4) \mapsto (e^{ik_1\theta_1} z_1, e^{ik_1\theta_1} z_2, e^{ik_1\theta_1} z_3, e^{ik_1\theta_1} z_4) \\
\theta_2 &: \ (z_1, z_2, z_3, z_4) \mapsto (e^{ik_2\theta_2} z_1, e^{ik_2\theta_2} z_2, e^{-ik_2\theta_2} z_3, e^{-ik_2\theta_2} z_4) \\
\theta_3 &: \ (z_1, z_2, z_3, z_4) \mapsto (e^{ik_3\theta_3} z_1, e^{-ik_3\theta_3} z_2, e^{ik_3\theta_3} z_3, e^{-ik_3\theta_3} z_4)
\end{aligned}$$

where k_1, k_2, k_3 are nonnegative integers and

- $z_1 = \bar{z}_4$ and $z_2 = \bar{z}_3$ if $k_1 = 0$.
- $z_1 = z_3$ and $z_2 = z_4$ if $k_2 = 0$.
- $z_1 = z_2$ and $z_3 = z_4$ if $k_3 = 0$.
- z_1, z_2, z_3, z_4 are any complex numbers otherwise.

The action of $K = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ may be written as

$$\begin{aligned}
\kappa_1 &: \ (z_1, z_2, z_3, z_4) \mapsto (\bar{z}_4, \bar{z}_3, \bar{z}_2, \bar{z}_1) \\
\kappa_2 &: \ (z_1, z_2, z_3, z_4) \mapsto (z_3, z_4, z_1, z_2) \\
\kappa_3 &: \ (z_1, z_2, z_3, z_4) \mapsto (z_2, z_1, z_4, z_3).
\end{aligned}$$

Note that bifurcating solutions have a well defined set of mode numbers $\underline{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$. These are induced by the action of $\mathbf{T}^3$ above and are associated with pattern formation that can be observed in experiments. Denote

$$\hat{W}_{\underline{k}} = \operatorname{span}\{z_1, z_2, z_3, z_4\}$$

and $W_{\underline{k}}$ the subspace fixed by K. Then we have that

$$\begin{aligned}
W_{\underline{k}} &= \{(\mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Re}(z_3), \mathrm{Re}(z_4)) | z_1 = z_2 = z_3 = z_4\} \\
&= \mathbb{R}.
\end{aligned}$$

Note that the representation $\hat{W}_{\underline{k}}$ of the group $K \dot{+} \mathbf{T}^3$ is a particular case of that used in chapter 2 in a slightly different setting: single mode bifurcations of reaction-diffusion equations with Neumann boundary conditions on n-dimensional rectangles. By finding an isomorphism between $\hat{W}_{\underline{k}}$ and $\ker L_r$ we will be able to use here all the results obtained in chapter 2.

Theorem 18 *Assume that the translation invariant solution $u = 0$ of $\Phi_r = 0$ undergoes a codimension one bifurcation with mode numbers $\underline{k}$ when the parameter R is increased from zero. Assume also that this occurs for some value of r such that $r_1 \neq r_2$. Then*

1. *In the extended problem, $\ker L_r$ is isomorphic to $\hat{W}_{\underline{k}}$.*
2. *If the boundary conditions (4.2) are imposed, $\ker L_r$ is spanned by $u_{\underline{k}}$ with components*

$$\begin{aligned}
v_1 &= a_{\underline{k}}[1] \sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3) \\
v_2 &= a_{\underline{k}}[2] \cos(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3) \\
v_3 &= a_{\underline{k}}[3] \cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3) \\
\Theta &= a_{\underline{k}}[4] \cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3) \\
p &= a_{\underline{k}}[5] \cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3),
\end{aligned}$$

where the $a_{\underline{k}}[j]$ are real numbers depending on the mode numbers and the unfolding parameters r.

Proof To prove part 1 we construct an isomorphism between the two representations of the group $K\dot{+}\mathbf{T}^3$. For that we need to introduce explicitly a set of generators for $\ker L_r$. In part 2 we restrict these generators to $\mathrm{Fix}(K)$.

1. We write z_1, z_2, z_3, z_4 as functions of the domain variables ξ_1, ξ_2, ξ_3. These functions will be used to generate the eigenmodes spanning $\ker L_r$. The condition for isomorphism is that acting on the z_j as above is the same as acting on the eigenfunctions as in section 4.3. Define

$$\begin{aligned} z_1 &: \quad (\xi_1,\xi_2,\xi_3) \longmapsto e^{i(k_1\xi_1+k_2\xi_2+k_3\xi_3)} \\ z_2 &: \quad (\xi_1,\xi_2,\xi_3) \longmapsto e^{i(k_1\xi_1+k_2\xi_2-k_3\xi_3)} \\ z_3 &: \quad (\xi_1,\xi_2,\xi_3) \longmapsto e^{i(k_1\xi_1-k_2\xi_2+k_3\xi_3)} \\ z_4 &: \quad (\xi_1,\xi_2,\xi_3) \longmapsto e^{i(k_1\xi_1-k_2\xi_2-k_3\xi_3)}. \end{aligned}$$

Let $\hat{\mathbf{u}}_k$ be an eigenmode in $\ker L_r$. Then we say that each component of $\hat{\mathbf{u}}_k$ belongs to a space spanned, over the reals, by the z_j as

$$\begin{aligned} v_1 &\in \mathrm{span}\{-iz_1, -iz_2, -iz_3, -iz_4\} \\ v_2 &\in \mathrm{span}\{-iz_1, -iz_2, iz_3, iz_4\} \\ v_3 &\in \mathrm{span}\{-iz_1, iz_2, -iz_3, iz_4\} \\ \Theta &\in \mathrm{span}\{-iz_1, iz_2, -iz_3, iz_4\} \\ p &\in \mathrm{span}\{z_1, z_2, z_3, z_4\} \end{aligned} \tag{4.5}$$

and checking that the two actions are isomorphic is a straightforward calculation.

2. In order to restrict $\ker L_r$ to $\mathrm{Fix}(K)$ it is more convenient to write (4.5) in the equivalent form

$$\begin{aligned} v_1(\xi) &\in \mathrm{span}\left\{z_1(\xi_1-\tfrac{\pi}{2},\xi_2,\xi_3), z_2(\xi_1-\tfrac{\pi}{2},\xi_2,\xi_3), z_3(\xi_1-\tfrac{\pi}{2},\xi_2,\xi_3), z_4(\xi_1-\tfrac{\pi}{2},\xi_2,\xi_3)\right\} \\ v_2(\xi) &\in \mathrm{span}\left\{z_1(\xi_1,\xi_2-\tfrac{\pi}{2},\xi_3), z_2(\xi_1,\xi_2-\tfrac{\pi}{2},\xi_3), z_3(\xi_1,\xi_2-\tfrac{\pi}{2},\xi_3), z_4(\xi_1,\xi_2-\tfrac{\pi}{2},\xi_3)\right\} \\ v_3(\xi) &\in \mathrm{span}\left\{z_1(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_2(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_3(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_4(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2})\right\} \\ \Theta(\xi) &\in \mathrm{span}\left\{z_1(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_2(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_3(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2}), z_4(\xi_1,\xi_2,\xi_3-\tfrac{\pi}{2})\right\} \\ p(\xi) &\in \mathrm{span}\left\{z_1(\xi_1,\xi_2,\xi_3), z_2(\xi_1,\xi_2,\xi_3), z_3(\xi_1,\xi_2,\xi_3), z_4(\xi_1,\xi_2,\xi_3)\right\}. \end{aligned}$$

The result follows by fixing these generators by the action of K in section 4.3.

□

By theorem 18, there is no loss of generality in working with the abstract representation W_k. For ease of exposition the most relevant results obtained in chapter 2 will be reproduced here when appropriate. However, we should not forget the aim of this chapter: to see how the abstract settings of the previous chapters apply to a concrete physical problem. This is why at some points we go back to see what is happening in the Bénard problem.

Before proceeding to bifurcations of higher codimension we denote

$$V_k = \mathrm{span}\{u_k\},$$

which, we recall, is $\ker L_r$ when the single mode $\underline{k}$ bifurcates from $u = 0$ and the boundary conditions (4.2) are imposed.

4.4.2 Simultaneous Bifurcation of Three Distinct Modes

Now assume that $r = r_c$ is such that, by increasing the parameter R from zero, we cross a critical value R_c, for which three distinct modes $\underline{k}, \underline{l}, \underline{m} \in \mathbb{N}^3$ bifurcate simultaneously from $u = 0$. Then, with the boundary conditions (4.2), we have

$$\ker L_r \;=\; V_k \oplus V_l \oplus V_m. \tag{4.6}$$

In the associated periodic boundary value problem, $\ker L_r$ is isomorphic to $\tilde{W}_k \oplus \tilde{W}_l \oplus \tilde{W}_m$ and $K\dot{+}\mathbf{T}^3$ acts on each irreducible block as in the previous section. We may assume, without loss of generality that

$$\mathrm{hcf}(k_j, l_j, m_j) = 1 \qquad \text{for} \qquad 1 \leq j \leq 3. \tag{4.7}$$

Otherwise we factor out the kernel of the group action and take it into account at the end when interpreting the results. Now (4.6) is isomorphic to

$$\mathrm{Fix}(K) \;=\; W_k \oplus W_l \oplus W_m = \mathbb{R}^3.$$

We define

$$\lambda \;=\; R - R_c$$

so that the bifurcation occurs at $\lambda = 0$. We want to know what are the symmetry constraints imposed on the bifurcation equations

$$f(x, y, z, \lambda) \;=\; 0,$$

where f maps $\mathbb{R}^3 \times \mathbb{R}$ onto $\mathbb{R}^3$.

As in chapter 2 we denote by N the subgroup $N_{K\dot{+}\mathbf{T}^3}(K)/K$, which is the normalizer of K in $K\dot{+}\mathbf{T}^3$ factored by K. The group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the mapping f must be equivariant under its action. As in section 2.5, the action of the normalizer N on $W_k \oplus W_l \oplus W_m$ is generated by translations of π along each of the directions chosen to generate $\mathbf{T}^3$

$$\begin{aligned}
\tau_1 :\ & (x, y, z) \mapsto ((-1)^{k_1} x, (-1)^{l_1} y, (-1)^{m_1} z) \\
\tau_2 :\ & (x, y, z) \mapsto ((-1)^{k_2} x, (-1)^{l_2} y, (-1)^{m_2} z) \\
\tau_3 :\ & (x, y, z) \mapsto ((-1)^{k_3} x, (-1)^{l_3} y, (-1)^{m_3} z).
\end{aligned} \tag{4.8}$$

Note that by assumption (4.7) all the τ_j act nontrivially and depending on the parities of the mode numbers, one or two of them may be redundant. Therefore we have that N factored by the kernel of its action is isomorphic to

- $\mathbf{Z}_2$ if one and only one of the τ_j is redundant.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ if two and only two of the τ_j are redundant.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ if all τ_j are nonredundant.

It can be shown that if N factored by the kernel of its action is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ then the mapping f is a generic N-equivariant and the form of the bifurcation equations is easy to obtain.

Theorem 19 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the three modes $\underline{k}, \underline{l}, \underline{m} \in N^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 \neq r_2$. If all r_j in (4.8) are nonredundant then the form of the reduced bifurcation equations is*

$$\begin{aligned} f_1(x,y,z,\lambda) = ax &= 0 \\ f_2(x,y,z,\lambda) = by &= 0 \\ f_3(x,y,z,\lambda) = cz &= 0 \end{aligned} \tag{4.9}$$

where a, b, c are functions of x^2, y^2, z^2 and λ.

Proof If all r_j are nonredundant, the problem has $Z_2 \oplus Z_2 \oplus Z_2$-symmetry. It is easy to see that the invariant functions are all the even functions in x, y, z. Then the equivariant mappings are obtained as in theorem 11, chapter 2 and the result follows. □

In appendix A the normal form of system (4.9) and its universal unfolding are given. Tables of nondegeneracy conditions and branching equations are obtained. In section 4.6, a Liapunov-Schmidt reduction is performed. This reduction gives the values of some low order derivatives of the coefficients a, b, c. In this case derivatives up to third order are enough. Then, the tables in the appendix will be used to obtain the bifurcation diagrams.

If the action of N has a nontrivial kernel the problem is a bit more subtle. In this case the periodic extension induces extra symmetries. In chapter 2 we deal with this problem in three steps:

1. Impose explicitly the constraints on T^3-invariant monomials.

2. Restrict these constraints to Fix(K) and use an algorithm to obtain a minimal set of generators for the restricted $K\dot{+}T^3$-invariants.

3. Generate the restricted $K\dot{+}T^3$-equivariants by using the result of 2 in theorem 11.

Then the restricted $K\dot{+}T^3$-equivariants are generated by the result of step 3 modulo the ring of invariants generated by the result of step 2.

4.5 Bifurcations with a $K\dot{+}S_2\dot{+}T^3$-Symmetric Extension

The notion of codimension depends on the symmetry of the problem and, by theorem 17, the condition $r_1 = r_2$ imposes an extra S_2-symmetry. Under this condition, the unfolding is being restricted to one parameter and codimension three bifurcations are no longer generic. In this diagonal of the r-space, bifurcations are expected to occur in regions as follows:

- Codimension one along lines.

- Codimension two at isolated points.

As in section 4.4 we begin with a brief description of codimension one bifurcations. More attention will be concentrated on simultaneous bifurcations of two distinct modes by increasing the Rayleigh number R from zero. In appendix B we unfold these bifurcations with the diagonal of the r-space and then break the $\mathbf{S}_2$-symmetry with a parameter transverse to this diagonal.

As in section 4.3, by reflecting across the boundaries any solution of the problem $\Phi_r = 0$ satisfying the boundary conditions (4.2) we construct another one consisting of the same equation and periodic boundary conditions on a larger domain. The second problem is invariant under an action of the group $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$. By theorem 17, the solutions satisfying the boundary conditions (4.2) are in one to one correspondence with those of the periodic boundary value problem that are fixed by the action of K.

4.5.1 Single Mode Bifurcations

Now the solution $u = 0$ undergoes a single mode bifurcation with $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-symmetry when $\ker L_r$ is an irreducible representation of this group. Assume that this holds for some r when the bifurcation parameter R crosses a critical value. As in section 4.4, bifurcating solutions have a well defined set of mode numbers $\underline{k} \in \mathbb{N}^3$. Let the group $\mathbf{S}_2$ act on the mode numbers as

$$s: \ (k_1, k_2, k_3) \mapsto (k_2, k_1, k_3).$$

By noting that $\mathbf{S}_2$ acts trivially if and only if $k_1 = k_2$, we construct an irreducible representation of $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$ as

- $\hat{W}_k$ if $k_1 = k_2$
- $\hat{W}_k \oplus \hat{W}_{sk}$ if $k_1 \neq k_2$

where $\hat{W}_k$ is as in section 4.4. On the other hand, when a solution with mode numbers $\underline{k}$ satisfying $\Phi_r = 0$ and the boundary conditions (4.2) bifurcates from $u = 0$ we have that

- $\ker L_r = V_k$ if $k_1 = k_2$
- $\ker L_r = V_k \oplus V_{sk}$ if $k_1 \neq k_2$.

As in section 4.4 it can be shown that $\ker L_r$ is isomorphic to

- $\mathrm{Fix}(K) = W_k = \mathbb{R}$ if $k_1 = k_2$
- $\mathrm{Fix}(K) = W_k \oplus W_{sk} = \mathbb{R}^2$ if $k_1 \neq k_2$

and we have the tools necessary to proceed with a more detailed study of higher codimension bifurcations.

4.5.2 Simultaneous Bifurcation of Two Distinct Modes

Assume that $r = r_c$ is such that, by increasing the parameter R, we cross a critical value R_c, for which two distinct modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ bifurcate simultaneously from $u = 0$. Then, if the boundary conditions (4.2) are imposed we have that

- $\ker L_r = V_k \oplus V_l$ if $k_1 = k_2$ and $l_1 = l_2$
- $\ker L_r = V_k \oplus V_l \oplus V_{sl}$ if $k_1 = k_2$ and $l_1 \neq l_2$
- $\ker L_r = V_k \oplus V_{sk} \oplus V_l \oplus V_{sl}$ if $k_1 \neq k_2$ and $l_1 \neq l_2$.

By analogy with the previous section we construct a direct sum of irreducible representations of the group $K\dot{+}S_2\dot{+}T^3$ for which the subspace fixed by K is

- $\mathrm{Fix}(K) = W_k \oplus W_l = \mathbb{R}^2$ if $k_1 = k_2$ and $l_1 = l_2$
- $\mathrm{Fix}(K) = W_k \oplus W_l \oplus W_{sl} = \mathbb{R}^3$ if $k_1 = k_2$ and $l_1 \neq l_2$
- $\mathrm{Fix}(K) = W_k \oplus W_{sk} \oplus W_l \oplus W_{sl} = \mathbb{R}^4$ if $k_1 \neq k_2$ and $l_1 \neq l_2$.

Note that as explained in section 4.4.2, there is no loss of generality in assuming that

$$\mathrm{hcf}(k_1, k_2, l_1, l_2) = 1$$
$$\mathrm{hcf}(k_3, l_3) = 1.$$

As in section 4.4 it can be shown that, when the boundary conditions (4.2) are imposed, $\ker L_r$ is isomorphic to $\mathrm{Fix}(K)$ as above. From now on, without loss of generality, we work with $\mathrm{Fix}(K)$ so that several results of chapters 2 and 3 apply directly. Once more we define

$$\lambda \;=\; R - R_c$$

so that the bifurcation is located at $\lambda = 0$. We want to know what are the symmetry constraints on the bifurcation equations

$$f(x, y, \lambda) \;=\; 0,$$

where f maps $\mathrm{Fix}(K) \times \mathbb{R}$ onto $\mathrm{Fix}(K)$.

By analogy with section 4.4 we denote by N the subgroup $N_{K\dot{+}S_2\dot{+}T^3}(K)/K$ and the mapping f must be equivariant under its action. As we will see below, f is not always a generic N-equivariant. In order to write explicitly the action of N on $\mathrm{Fix}(K)$ we need to consider separately three distinct cases.

(a) $k_1 = k_2$ and $l_1 = l_2$

This case reduces to that of 3-dimensional rectangles described in section 2.4. The results are reproduced here to make this chapter complete. The group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and its action on $W_k \oplus W_l$ is generated by translations of π along each of the directions chosen to generate $\mathbf{T}^3$

$$\begin{aligned} \tau_1 : &\quad (x, y) \mapsto ((-1)^{k_1}x, (-1)^{l_1}y) \\ \tau_3 : &\quad (x, y) \mapsto ((-1)^{k_3}x, (-1)^{l_3}y). \end{aligned}$$

The translation τ_2 is omitted because it acts as τ_1. Note that all τ_j act nontrivially and some of them may be redundant. In fact, as in theorem 5, we have that N factored by the kernel of its action is isomorphic to

- $\mathbf{Z}_2$ if all k_j have the same parity and all l_j have the same parity.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ otherwise.

We proceed by giving a general form for the bifurcation equations.

Theorem 20 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 = r_2$. If $k_1 = k_2$ and $l_1 = l_2$ then the form of the reduced bifurcation equations is*

1. *If all k_j have the same parity and all l_j have the same parity*

$$\begin{aligned} f_1(x, y, \lambda) = ax + cx^{l-1}y^k &= 0 \\ f_2(x, y, \lambda) = by + dx^l y^{k-1} &= 0 \end{aligned}$$

where $k = max_j k_j$ and $l = max_j l_j$.

2. *Otherwise*

$$\begin{aligned} f_1(x, y, \lambda) = ax &= 0 \\ f_2(x, y, \lambda) = by &= 0 \end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ.

Proof See section 2.4. □

(b) $k_1 = k_2$ and $l_1 \neq l_2$

In this case the group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_4$ and its action on $W_k \oplus W_l \oplus W_{sl}$ is generated by

$$\begin{aligned} \tau_1 : &\quad (x, y_1, y_2) \mapsto ((-1)^{k_1}x, (-1)^{l_1}y_1, (-1)^{l_2}y_2) \\ s : &\quad (x, y_1, y_2) \mapsto (x, y_2, y_1) \\ \tau_3 : &\quad (x, y_1, y_2) \mapsto ((-1)^{k_3}x, (-1)^{l_3}y_1, (-1)^{l_3}y_2). \end{aligned} \tag{4.10}$$

Note that the translation τ_2 is omitted because it acts as $s\tau_1 s$. As before, one of these generators may be redundant. By inspection of (4.10) we see that N factored by the kernel of its action is isomorphic to

- $\mathbf{Z}_2 \oplus \mathbf{S}_2$ if all k_j have the same parity and all l_j have the same parity
  or k_1, k_3+1 have the same parity and l_1, l_2, l_3+1 have the same parity.

- $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{S}_2$ if all k_j have the same parity and l_1, l_2, l_3+1 have the same parity
  or k_1, k_3+1 have the same parity and all l_j have the same parity.

- $\mathbf{D}_4$ if l_1, l_2+1 have the same parity and k_3 is even.

- $\mathbf{Z}_2 \oplus \mathbf{D}_4$ if l_1, l_2+1 have the same parity and k_3 is odd.

It can be shown that if N factored by the kernel of its action is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_4$ then the mapping f is a generic N-equivariant and the form of the bifurcation equations is easily obtained.

Theorem 21 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 = r_2$. If l_1, l_2 have different parities and k_3 is odd then the form of the reduced bifurcation equations is*

$$\begin{aligned} f_1(x, y_1, y_2, \lambda) &= ax = 0 \\ f_2(x, y_1, y_2, \lambda) &= (b + c\delta)y_1 = 0 \\ f_3(x, y_1, y_2, \lambda) &= (b - c\delta)y_2 = 0 \end{aligned} \tag{4.11}$$

where a, b, c are functions of x^2, $y_1^2 + y_2^2$, $(y_2^2 - y_1^2)^2$ and λ; and $\delta = y_2^2 - y_1^2$.

Proof The action of $\mathbf{Z}_2$ generated by τ_2 implies that f_1 is odd in x and f_2, f_3 are even in x. Terms involving y_1, y_2 are determined by the action of $\mathbf{D}_4$ generated by s, τ_3. See Golubitsky *et al.* [11], chapter XVII, section 4. □

The normal form for system (4.11) together with the universal unfolding that keeps the symmetry are given in appendix B. Tables of nondegeneracy conditions, branching equations and some bifurcation diagrams are obtained. Then the $\mathbf{S}_2$ symmetry is broken with another unfolding parameter.

If the action of N has a nontrivial kernel, a general form for the bifurcation equations cannot be given. The method for finding the generators of the restricted invariants and equivariants splits into five steps:

1. Impose explicitly the constraints on $\mathbf{T}^3$-invariant monomials.

2. Restrict these constraints to $\mathrm{Fix}(K)$ and use an algorithm to obtain a minimal set of generators for the restricted $\mathbf{T}^3$-invariants.

3. Generate the restricted $\mathbf{T}^3$-equivariants by using the result of 2 in theorem 11.

4. Symmetrize the result of 2 over $\mathbf{S}_2$ to obtain the restricted $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-invariants.

5. Symmetrize the result of 3 over $\mathbf{S}_2$ to obtain the restricted $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-equivariants.

Then the restricted $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-equivariants are generated by the result of step 5 modulo the ring of invariants generated by the result of step 4. Now we can justify the statement above that a general form for the bifurcation equations cannot be given: the method for obtaining them involves an algorithm. Thus, we have to work case by case if the action of the group N has a nontrivial kernel.

(c) $k_1 \neq k_2$ and $l_1 \neq l_2$

In this case the group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_4$ and its action on $W_k \oplus W_{sk} \oplus W_l \oplus W_{sl}$ is generated by

$$\begin{aligned} r_1 &: \ (x_1, x_2, y_1, y_2) \mapsto ((-1)^{k_1} x_1, (-1)^{k_2} x_2, (-1)^{l_1} y_1, (-1)^{l_2} y_2) \\ s &: \ (x_1, x_2, y_1, y_2) \mapsto (x_2, x_1, y_2, y_1) \\ r_2 &: \ (x_1, x_2, y_1, y_2) \mapsto ((-1)^{k_2} x_1, (-1)^{k_1} x_2, (-1)^{l_2} y_1, (-1)^{l_1} y_2). \end{aligned}$$

As before, N factored by the kernel of its action is isomorphic to some normal subgroup of $\mathbf{Z}_2 \oplus \mathbf{D}_4$ containing $\mathbf{S}_2$. Whatever parities the mode numbers have, a general form for the bifurcation equations cannot be given: we need to follow the steps 1 to 5 as in the previous case to compute the generators of the restricted invariants and equivariants.

4.6 Some Generalities About the Liapunov-Schmidt Reduction

This section gives a description of the Liapunov-Schmidt reduction as in Golubitsky and Schaeffer [30]. Then we apply it to steady-state bifurcations of the family of operators Φ_r parametrized by $r \in \mathbb{R}^2$ satisfying the boundary conditions (4.2). This procedure projects the dynamics onto the kernel of the linearized operator L_r. Denote

$$\begin{aligned} \mathcal{X} &= (C^2(\Omega))^5 \\ \mathcal{Y} &= (C^0(\Omega))^5. \end{aligned}$$

The operator

$$\begin{aligned} \Phi : \mathcal{X} \times \mathbb{R} \times \mathbb{R}^2 &\to \mathcal{Y} \\ (u, R, r) &\mapsto \Phi_r(u, R) \end{aligned}$$

in section 4.2 is a smooth C^∞ mapping and R_c, r_c are chosen such that the derivative $L = (d\Phi)_{(0,R_c,r_c)}$ is a Fredholm operator of index 0. Substituting

$$\begin{aligned} r &\mapsto \rho + r_c \\ R &\mapsto \lambda + R_c \end{aligned}$$

we say that a bifurcation for the parameter λ occurs at $\rho = 0$ and $\lambda = 0$ if L has a nontrivial kernel, which will be assumed later in all applications. Because L is Fredholm, it has a finite-dimensional kernel. We choose a basis for ker L and write

$$\ker L = \operatorname{span}\{u_1, \ldots, u_n\}.$$

Also, range L has finite codimension and we may write

$$(\text{range } L)^{\perp} = \ker L^*,$$

where L^* is the adjoint of L. Now, L having index 0 means that ker L and ker L^*have the same dimension. So we choose a basis for ker L^* and write

$$\ker L^* = \text{span}\{u_1^*, \ldots, u_n^*\}.$$

We may decompose

$$\mathcal{X} = \ker L \oplus M \tag{4.12}$$
$$\mathcal{Y} = N \oplus \text{range } L. \tag{4.13}$$

Let E be the projection of $\mathcal{Y}$ onto range L and write the equations $\Phi(u, \lambda, \rho) = 0$ in the equivalent form

$$E\Phi(u, \lambda, \rho) = 0 \tag{4.14}$$
$$(I - E)\Phi(u, \lambda, \rho) = 0 \tag{4.15}$$

where I is the identity on $\mathcal{Y}$. Note that $I - E$ is the projection of $\mathcal{Y}$ onto $N = \ker L^*$. By decomposition (4.12) we may write any $u \in \mathcal{X}$ as

$$u = \sum_{j=1}^{n} x_j u_j + w$$

where $w \in M$. Denote $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $U = (u_1, \ldots, u_n)$. By the implicit function theorem, equations (4.14) may be solved as a function $w = W(x, \lambda, \rho)$. Thus there exists a function

$$W : \ker L \times \mathbb{R} \to M$$

such that

$$E\Phi(x.U + W(x, \lambda, \rho), \lambda, \rho) \equiv 0.$$

Substituting u we make the projection I-E by using the basis chosen for ker L and get the so called Liapunov-Schmidt reduced mapping

$$f : \ker L \times \mathbb{R} \times \mathbb{R}^2 \to \ker L^*$$

as

$$f_j(x, \lambda, \rho) = \langle u_j^*, \Phi(x.U + W(x, \lambda, \rho), \lambda, \rho)\rangle,$$

for $1 \leq j \leq n$. Note that the formula for f is not explicit: it depends on the implicitly defined function W. However, the derivatives of W can be expressed in terms of Φ and we can compute terms of the Taylor expansion of f. By using the symmetry constraints described in the previous sections we see which terms of the expansion we need in order to determine the bifurcation equations.

4.7 Basic Tools

The aim of this section is to extract some information from the Boussinesq equations (4.1) with boundary conditions (4.2) so that we will be ready to perform Liapunov-Schmidt reductions in the following sections. In section 4.7.1 we locate bifurcation points in the parameter space. Section 4.7.2 is concerned with symmetries of the linearized operator L_r that will simplify the calculations.

4.7.1 Location of Primary Bifurcations

Here we identify parameter values for which there is a bifurcation from the solution $u = 0$ of Φ, satisfying the boundary conditions (4.2). So we want the values of R such that $L_r(u, R)$ has a nontrivial kernel. By theorem 18, associated to any bifurcating solution is a set of mode numbers $k = (k_1, k_2, k_3) \in \mathbb{N}^3$ and an eigenmode u_k with components

$$\begin{aligned}
v_1 &= a_k[1] \sin(k_1\xi_1) \cos(k_2\xi_2) \cos(k_3\xi_3) \\
v_2 &= a_k[2] \cos(k_1\xi_1) \sin(k_2\xi_2) \cos(k_3\xi_3) \\
v_3 &= a_k[3] \cos(k_1\xi_1) \cos(k_2\xi_2) \sin(k_3\xi_3) \\
\Theta &= a_k[4] \cos(k_1\xi_1) \cos(k_2\xi_2) \sin(k_3\xi_3) \\
p &= a_k[5] \cos(k_1\xi_1) \cos(k_2\xi_2) \cos(k_3\xi_3),
\end{aligned}$$

where the $a_k[j]$ are real numbers depending on the mode numbers and the unfolding parameters r. As in section 4.4, we denote

$$V_k = \operatorname{span}\{u_k\}.$$

Let r be any point in the unfolding parameter space. By solving

$$L_r(u_k, R_k) = 0, \tag{4.16}$$

we calculate the critical Rayleigh number R_k and the coefficients a_k of the eigenmode. By denoting

$$\Lambda_k = \frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2} + k_3^2,$$

the solution of (4.16) is

$$R_k = \frac{\Lambda_k^3}{\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2}},$$

and

$$\begin{aligned}
a_k[1] &= \frac{k_1}{r_1}(\Lambda_k^2 - R_k) \\
a_k[2] &= \frac{k_2}{r_2}(\Lambda_k^2 - R_k) \\
a_k[3] &= k_3\Lambda_k^2 \\
a_k[4] &= k_3\Lambda_k R_k \\
a_k[5] &= \Lambda_k(\Lambda_k^2 - R_k).
\end{aligned}$$

As a simplification of notation, the unfolding parameter r has not been kept as a subscript. However, we remark the dependence of a_k and u_k on r. From now the dependence on r should be seen implicitly in the mode numbers.

As a tool for the Liapunov-Schmidt reductions in later sections we have that when $R = R_k$, the function u_k^* with components

$$
\begin{aligned}
v_1^* &= \frac{k_1}{r_1}\sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)\\
v_2^* &= \frac{k_2}{r_2}\cos(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3)\\
v_3^* &= k_3\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
\Theta^* &= (R_k - \Lambda_k^2)\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
p^* &= k_3\cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)
\end{aligned}
$$

is an eigenmode of the adjoint operator L_r^*. We keep also the notation

$$V_k^* = \operatorname{span}\{u_k^*\}.$$

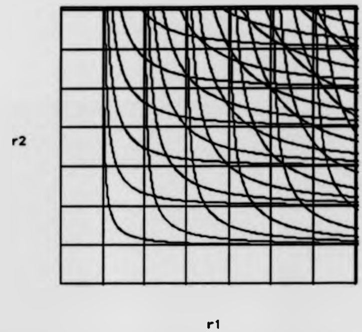
The graph of R_k is a folded surface with minimum

$$R_k^{min} = \frac{27}{4}k_3^4 \tag{4.17}$$

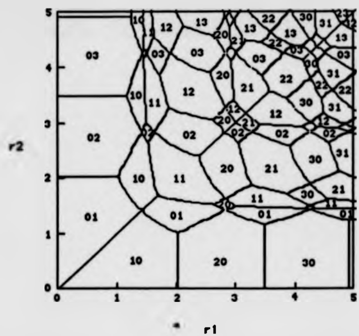
along the curve

$$\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2} = \frac{1}{2}k_3^2. \tag{4.18}$$

From (4.17) we see that R_k^{min} increases very fast with k_3. From (4.18) we get that $k_1 = k_2 = 0$ when $k_3 = 0$. Thus, the natural candidates for primary bifurcations by increasing the parameter R from zero are modes such that $k_3 = 1$. For such sets of mode numbers, the curves R_k^{min} are projected in the 2-dimensional unfolding parameter space as



The unfolding parameter space is divided into regions according to the mode that bifurcates by increasing the Rayleigh number from zero. The result, obtained numerically, is as



4.7.2 $\mathbf{S}_2$-Equivariance of the Linearized Operator

Let the group $\mathbf{S}_2$ act on the unfolding parameters r and $\mathcal{X}$ as in previous section. By making the only generator s act on the set of mode numbers k as

$$s: \quad (k_1, k_2, k_3) \mapsto (k_2, k_1, k_3)$$

we consider implicitly an action of s on r. We observe that the solution u_k of $L_r(u) = 0$ calculated in section 4.7.1 is $\mathbf{S}_2$-equivariant

$$u_{sk}(s\xi) = su_k(\xi).$$

This suggests another result concerning a symmetry of the linear operator L_r. Let the group $\mathbf{S}_2$ act on the operator L_r as

$$s: \quad (L_r[1], L_r[2], L_r[3], L_r[4], L_r[5]) \mapsto (L_r[2], L_r[1], L_r[3], L_r[4], L_r[5]).$$

This group action is the basis for the proof of the following

Lemma 12 *Let $w \in (ker\ L_r)^\perp$ be a solution of $L_r(u) = b$ where $b \in \mathcal{Y}$ is $\mathbf{S}_2$-equivariant. Then w is $\mathbf{S}_2$-equivariant.*

Proof The equation $L_r(u) = b$ has a unique solution in $(\ker L_r)^\perp$. Let $w(\xi)$ be this solution

$$L_r(w(\xi)) = b(\xi). \tag{4.19}$$

We claim that $sw(s\xi)$ is a solution of the same equation and by uniqueness it must be equal to $w(\xi)$. This gives the equivariance that we want

$$w(s\xi) = sw(\xi).$$

To prove the claim we observe that

$$L_{sr}(sw(s\xi)) = b(s\xi). \tag{4.20}$$

Now a direct calculation shows that the operator L_r commutes with the action of $\mathbf{S}_2$ as

$$L_{sr}(sw(s\xi)) = L_r(w(\xi))$$

and by equivariance of $b(\xi)$ we have that

$$b(s\xi) = sb(\xi)$$

where s acts on b by permuting the first two components. Therefore, equation (4.20) is the same as (4.19) and $sw(s\xi)$ is a solution. □

4.8 Interaction of Three Modes in a Box with Rectangular Cross Section

Assume that $r = r_c$ is such that three solutions with mode numbers $\underline{k}, \underline{l}, \underline{m} \in \mathbb{N}^3$ bifurcate simultaneously when the Rayleigh number is increased across a critical value $R = R_c$. As in section 4.7 we have that ker L and ker L^* are 3-dimensional spaces as

$$\begin{aligned} \ker L &= V_k \oplus V_l \oplus V_m \\ \ker L^* &= V_k^* \oplus V_l^* \oplus V_m^*. \end{aligned}$$

Let

$$\begin{aligned} \underline{k} &= (0,1,1) \\ \underline{l} &= (1,1,1) \\ \underline{m} &= (2,0,1). \end{aligned}$$

A calculation shows that these modes bifurcate simultaneously at two points in the unfolding parameter space. These points are approximately as

	Critical point 1	Critical point 2
r_{1c}	2.634	2.992
r_{2c}	1.521	1.496

and the critical values of the bifurcation parameter are approximately

	Critical point 1	Critical point 2
R_c	6.797	6.778

The Liapunov-Schmidt calculations will be performed for the two points in parallel since they both satisfy the requirements of the procedure described in the sequel. As in section 4.4.2, the bifurcation equations obtained by a Liapunov-Schmidt reduction have $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry. Thus, up to 3rd order they are of the form

$$\begin{aligned} f_1 &= (a_{N_1}N_1 + a_{N_2}N_2 + a_{N_3}N_3 + a_\lambda \lambda + a_{\rho_1}\rho_1 + a_{\rho_2}\rho_2)x = 0 \\ f_2 &= (b_{N_1}N_1 + b_{N_2}N_2 + b_{N_3}N_3 + b_\lambda \lambda + b_{\rho_1}\rho_1 + b_{\rho_2}\rho_2)y = 0 \\ f_3 &= (c_{N_1}N_1 + c_{N_2}N_2 + c_{N_3}N_3 + c_\lambda \lambda + c_{\rho_1}\rho_1 + c_{\rho_2}\rho_2)z = 0 \end{aligned}$$

where

$$N_1 = x^2 \qquad N_2 = y^2 \qquad N_3 = z^2$$

and

$$\lambda = R - R_c \qquad \rho_1 = r_1 - r_{1_c} \qquad \rho_2 = r_2 - r_{2c}.$$

According to Golubitsky and Schaeffer [30] the coefficients are

$$
\begin{aligned}
a_{N_1} &= \frac{1}{6}\langle u_k^*, V_{kk}\rangle \\
a_{N_2} &= \frac{1}{2}\langle u_k^*, V_{kl}\rangle \\
a_{N_3} &= \frac{1}{2}\langle u_k^*, V_{km}\rangle \\
a_\lambda &= \langle u_k^*, (d\Phi_\lambda).u_k - d^2\Phi(u_k, L^{-1}E\Phi_\lambda)\rangle \\
a_{\rho_j} &= \langle u_k^*, (d\Phi_{\rho_j}).u_k - d^2\Phi(u_k, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2 \\
b_{N_1} &= \frac{1}{2}\langle u_l^*, V_{lk}\rangle \\
b_{N_2} &= \frac{1}{6}\langle u_l^*, V_{ll}\rangle \\
b_{N_3} &= \frac{1}{2}\langle u_l^*, V_{lm}\rangle \\
b_\lambda &= \langle u_l^*, (d\Phi_\lambda).u_l - d^2\Phi(u_l, L^{-1}E\Phi_\lambda)\rangle \\
b_{\rho_j} &= \langle u_l^*, (d\Phi_{\rho_j}).u_l - d^2\Phi(u_l, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2 \\
c_{N_1} &= \frac{1}{6}\langle u_m^*, V_{mk}\rangle \\
c_{N_2} &= \frac{1}{2}\langle u_m^*, V_{ml}\rangle \\
c_{N_3} &= \frac{1}{2}\langle u_m^*, V_{mm}\rangle \\
c_\lambda &= \langle u_m^*, (d\Phi_\lambda).u_m - d^2\Phi(u_m, L^{-1}E\Phi_\lambda)\rangle \\
c_{\rho_j} &= \langle u_m^*, (d\Phi_{\rho_j}).u_m - d^2\Phi(u_m, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2.
\end{aligned}
$$

where E is the projection onto range L and, taking into account that the initial equations have only quadratic nonlinearity, we have that

$$
\begin{aligned}
V_{kk} &= -3d^2\Phi(u_k, w_{kk}) \\
V_{kl} &= -d^2\Phi(u_k, w_{ll}) - 2d^2\Phi(u_l, w_{kl}) \\
V_{km} &= -d^2\Phi(u_k, w_{mm}) - 2d^2\Phi(u_m, w_{km}) \\
V_{lk} &= -d^2\Phi(u_l, w_{kk}) - 2d^2\Phi(u_k, w_{kl}) \\
V_{ll} &= -3d^2\Phi(u_l, w_{ll}) \\
V_{lm} &= -d^2\Phi(u_l, w_{mm}) - 2d^2\Phi(u_m, w_{lm}) \\
V_{mk} &= -d^2\Phi(u_m, w_{kk}) - 2d^2\Phi(u_k, w_{km}) \\
V_{ml} &= -d^2\Phi(u_m, w_{ll}) - 2d^2\Phi(u_l, w_{lm}) \\
V_{mm} &= -3d^2\Phi(u_m, w_{mm})
\end{aligned}
$$

where

$$
\begin{aligned}
w_{kk} &= L^{-1}Ed^2\Phi(u_k, u_k) \\
w_{kl} &= L^{-1}Ed^2\Phi(u_k, u_l)
\end{aligned}
$$

$$\begin{aligned}
w_{km} &= L^{-1}Ed^2\Phi(u_k, u_m)\\
w_{ll} &= L^{-1}Ed^2\Phi(u_l, u_l)\\
w_{lm} &= L^{-1}Ed^2\Phi(u_l, u_m)\\
w_{mm} &= L^{-1}Ed^2\Phi(u_m, u_m).
\end{aligned}$$

4.8.1 Calculation of $a_{N_1}, b_{N_2}, c_{N_3}$

In this section we give a detailed description of the calculation of a_{N_1}. The coefficients b_{N_2} and c_{N_3} are obtained immediatly by substituting the mode numbers $\underline{k}$ by $\underline{l}$ and $\underline{m}$ respectively. The main technical difficulty is the calculation of w_{kk}. We reserve part (a) to deal with this problem. In part (b) we explain how to use Maple to do the rest of the work.

(a) Calculation of w_{kk}

The equation to solve in order to calculate w_{kk} is

$$L(w_{kk}) = Ed^2\Phi(u_k, u_k). \tag{4.21}$$

In order to simplify the notations we denote

$$\begin{aligned}
A_k &= \sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)\\
B_k &= \cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
C_k &= \sin(k_1\xi_1)\sin(k_2\xi_2)\sin(k_3\xi_3)\\
D_k &= \cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)\\
E_k &= \sin(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3)\\
F_k &= \sin(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3).
\end{aligned}$$

Recall from section 4.7 that

$$u_k = (a_k[1]A_k, a_k[2]A_{sk}, a_k[3]B_k, a_k[4]B_k, a_k[5]D_k)$$

where after calculating a_k explicitly we observed the equivariance under the group $\mathbf{S}_2$ as

$$a_{sk} = sa_k.$$

Now we have

$$\begin{aligned}
d^2\Phi[1](u_k, u_k) &= -\frac{2a_k[1]}{\sigma}\left(\frac{k_1}{r_1}a_k[1]A_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}E_k - k_3a_k[3]B_kF_k\right)\\
d^2\Phi[2](u_k, u_k) &= -\frac{2a_k[2]}{\sigma}\left(\frac{k_2}{r_2}a_k[2]A_{sk}D_k - \frac{k_1}{r_1}a_k[1]A_kE_k - k_3a_k[3]B_kF_{sk}\right)\\
d^2\Phi[3](u_k, u_k) &= -\frac{2a_k[3]}{\sigma}\left(k_3a_k[3]B_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}F_{sk} - \frac{k_1}{r_1}a_k[1]A_kF_k\right)\\
d^2\Phi[4](u_k, u_k) &= -2a_k[4]\left(k_3a_k[3]B_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}F_{sk} - \frac{k_1}{r_1}a_k[1]A_kF_k\right)\\
d^2\Phi[5](u_k, u_k) &= 0.
\end{aligned}$$

A direct calculation shows that

$$(u_j^*, d^2\Phi(u_k, u_k)) = 0$$

for $j = k, l, m$, which means that $d^2\Phi(u_k, u_k) \in$ range L. Recalling that E in equation (4.21) denotes an orthogonal projection onto range L, this equation is the same as

$$L(w_{kk}) = d^2\Phi(u_k, u_k). \tag{4.22}$$

Also by direct calculation we have that $d^2\Phi(u_k, u_k)$ is $\mathbf{S}_2$-equivariant. Thus, by lemma 12, the solution w_{kk} of system (4.22) must be such that

$$w_{sksk}(s\xi) = sw_{kk}(\xi). \tag{4.23}$$

Maple does not seem to solve PDEs and system (4.22) is very hard to solve without the help of some computer algebra. The approach we take here is to write w_{kk} as a polynomial function of $A_k, \ldots, F_k$. The coefficients in this polynomial are the solution of a system of linear algebraic equations. Denote

$$\begin{aligned}
X_k[1] &= (A_k D_k, A_{sk} E_k, B_k F_k, C_k F_{sk}) \\
X_k[2] &= (A_{sk} D_k, A_k E_k, B_k F_{sk}, C_k F_k) \\
X_k[3] &= (B_k D_k, A_{sk} F_{sk}, A_k F_k, C_k E_k) \\
X_k[4] &= (B_k D_k, A_{sk} F_{sk}, A_k F_k, C_k E_k) \\
X_k[5] &= (A_k^2, A_{sk}^2, B_k^2, C_k^2, D_k^2, E_k^2, F_k^2, F_{sk}^2)
\end{aligned}$$

and note that $d^2\Phi[j](u_k, u_k)$ is a linear combination of the components of $X_k[j]$. More precisely, by denoting

$$\begin{aligned}
b_k[1] &= \frac{2a_k[1]}{\sigma}\left(-\frac{k_1}{r_1}a_k[1], \frac{k_2}{r_2}a_k[2], k_3 a_k[3], 0\right) \\
b_k[2] &= \frac{2a_k[2]}{\sigma}\left(-\frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], k_3 a_k[3], 0\right) \\
b_k[3] &= \frac{2a_k[3]}{\sigma}\left(-k_3 a_k[3], \frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], 0\right) \\
b_k[4] &= 2a_k[4]\left(-k_3 a_k[3], \frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], 0\right) \\
b_k[5] &= (0,0,0,0,0,0,0,0),
\end{aligned}$$

we have that

$$d^2\Phi[j](u_k, u_k) = b_k[j].X_k[j].$$

Writing $w_{kk}[j]$ linearly on the components of $X_k[j]$ and substituting on (4.22) we get a system of linear algebraic equations on the coefficients. By doing this we arrived at the conclusion that the coordinates X_k are not convenient: the system we got is not determined and Maple cannot solve it. In order to change coordinates we introduce the matrices

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} P_1 & P_1 \\ P_1 & -P_1 \end{pmatrix}$$

A better coordinate system is

$$\begin{aligned} Y_k[j] &= P_1 X_k[j] \quad \text{for } 1 \le j \le 4, \\ Y_k[5] &= P_2 X_k[5]. \end{aligned}$$

Note that this change of coordinates is orthogonal. In fact we have that

$$P_1^{-1} = \frac{1}{4}P_1 \quad \text{and} \quad P_2^{-1} = \frac{1}{8}P_2.$$

Let

$$\begin{aligned} d^2\Phi[j](u_k, u_k) &= q_k[j].Y_k[j] \\ w_{kk}[j] &= p_k[j].Y_k[j], \end{aligned} \tag{4.24}$$

where

$$q_k[j] = P_1^{-1} b_k[j]$$

and p_k are the coefficients that we want to calculate. As another symmetry observation we have that Y_k is $\mathbf{S}_2$-equivariant as

$$\begin{aligned} Y_{sk}[1] &= Y_k[2] \\ Y_{sk}[3,2] &= Y_k[3,3] \\ Y_{sk}[4,2] &= Y_k[4,3] \\ Y_{sk}[5,2] &= -Y_k[5,8] \\ Y_{sk}[5,4] &= -Y_k[5,6], \end{aligned} \tag{4.25}$$

where $Y_k[i,j]$ represents the jth component of $Y_k[i]$. A simple reasoning says that w_{kk} written in the coordinates Y_k satisfies the equivariance (4.23) if and only if the coefficients p_k are constrained by the symmetries (4.25). For the same reason q_k satisfies the same symmetries.

Substituting (4.24) in system (4.22) we get a system of linear algebraic equations in the form

$$L_k p_k = q_k,$$

where q_k is already known and by permuting rows and columns, the matrix L_k may be written as

$$L_k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_k[1] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{sk}[1] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_k[2] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_k[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_k[4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{sk}[4] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_k[5] \end{pmatrix},$$

where the matrices along the diagonal are

$$\begin{aligned}
L_k[1] &= \begin{pmatrix} -4\frac{k_2^2}{r_2^2} & 4\frac{k_2}{r_2} \\ \frac{k_2}{r_2} & 0 \end{pmatrix} \\
L_k[2] &= \begin{pmatrix} -4\left(\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2}\right) & 0 & -4\frac{k_1}{r_1} \\ 0 & -4\left(\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2}\right) & -4\frac{k_2}{r_2} \\ -\frac{k_1}{r_1} & -\frac{k_2}{r_2} & 0 \end{pmatrix} \\
L_k[3] &= \begin{pmatrix} -4k_3^2 & 1 & 4k_3 \\ R & -4k_3^2 & 0 \\ k_3 & 0 & 0 \end{pmatrix} \\
L_k[4] &= \begin{pmatrix} -4\left(\frac{k_2^2}{r_2^2} + k_3^2\right) & 0 & 0 & 4\frac{k_2}{r_2} \\ 0 & -4\left(\frac{k_2^2}{r_2^2} + k_3^2\right) & 1 & 4k_3 \\ 0 & R & -4\left(\frac{k_2^2}{r_2^2} + k_3^2\right) & 0 \\ \frac{k_2}{r_2} & k_3 & 0 & 0 \end{pmatrix} \\
L_k[5] &= \begin{pmatrix} -4\Lambda_k & 0 & 0 & 0 & -4\frac{k_1}{r_1} \\ 0 & -4\Lambda_k & 0 & 0 & -4\frac{k_2}{r_2} \\ 0 & 0 & -4\Lambda_k & 1 & -4k_3 \\ 0 & 0 & R & -4\Lambda_k & 0 \\ -\frac{k_1}{r_1} & -\frac{k_2}{r_2} & -k_3 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Letting c_k be the corresponding reordering of p_k we have that

$$\begin{aligned}
c_k[0] &= p_k[5,1] \\
c_k[1] &= (p_k[2,1], p_k[5,2]) \\
c_k[2] &= (p_k[1,2], p_k[2,2], p_k[5,7]) \\
c_k[3] &= (p_k[3,1], p_k[4,1], p_k[5,3]) \\
c_k[4] &= (p_k[2,3], p_k[3,2], p_k[4,2], p_k[5,4]) \\
c_k[5] &= (p_k[1,4], p_k[2,4], p_k[3,4], p_k[4,4], p_k[5,5]),
\end{aligned}$$

and we let d_k be the same reordering of q_k. Now systems of the form

$$L_{sk}[j]c_{sk}[j] = d_{sk}[j]$$

are redundant since $c_{sk}[j]$ can be obtained by conjugacy of $c_k[j]$. So we are restricted to solve the systems

$$L_k[j]c_k[j] = d_k[j]$$

for $1 \le j \le 5$, which may be easily solved by hand. We may also ask Maple to solve these systems without worrying about any technical problem since the determinants are nonzero. All coefficients are now determined apart from $p_k[5,1]$ which can have any value. This is not surprising if we see that

$$Y_k[5,1] = 1.$$

Thus when we differentiate w_{kk} with respect to any ξ_j, the coefficient $p_k[5,1]$ will not appear. Finally we have w_{kk} and we can proceed with the rest of the calculation of a_{N_1}.

(b) The rest of the job with Maple

Now, Maple has no problem in doing the rest of the job. Recall that what is left is the computation of

$$V_{kk} = -3d^2\Phi(u_k, w_{kk}),$$

where u_k is one of the eigenfunctions generating ker L and w_{kk} is what we have just calculated. Given the values of u_k and w_{kk}, that we know at this point, Maple computes V_{kk} according to the formula above. Then we have one of the coefficients of the reduced equations by computing

$$\begin{aligned} a_{N_1} &= \frac{1}{6}(u_k^*, V_{kk}) \\ &= \frac{1}{6}\int_0^\pi \int_0^\pi \int_0^\pi u_k^* \cdot V_{kk}\,dx\,dy\,dz, \end{aligned}$$

where the dot is the usual dot product in $\mathbb{R}^5$ and we recall that u_k^* is one of the eigenfunctions generating ker L^*. Two of the remaining coefficients of the bifurcation equations are obtained by substituting the mode numbers $\underline{k}$ in the formulae above by l and $\underline{m}$ respectively. The approximate results are as follows

	Critical point 1	Critical point 2
a_{N_1}	$-\frac{2.712}{\sigma}\pi^3$	$-\frac{2.816}{\sigma}\pi^3$
b_{N_2}	$-\frac{1.529\sigma-0.104}{\sigma^2}\pi^3$	$-\frac{1.451\sigma-0.076}{\sigma^2}\pi^3$
c_{N_3}	$-\frac{4.148}{\sigma}\pi^3$	$-\frac{2.816}{\sigma}\pi^3$

4.8.2 Calculation of $a_{N_2}, a_{N_3}, b_{N_1}, b_{N_3}, c_{N_1}, b_{N_2}$

This section gives a detailed explanation of how to obtain a_{N_2}. The other five coefficients are obtained automatically by substitution of mode numbers. Again we reserve part (a) to deal with the main technical difficulty: the calculation of w_{kl}. In part (b) we show how Maple can do the rest of the work.

(a) Calculation of w_{kl}

Now we want w_{kl} satisfying the equation

$$L(w_{kl}) \;=\; Ed^2\Phi(u_k, u_l). \tag{4.26}$$

Recall that

$$\begin{aligned} u_k &= (a_k[1]A_k, a_k[2]A_{sk}, a_k[3]B_k, a_k[4]B_k, a_k[5]D_k) \\ u_l &= (a_l[1]A_l, a_l[2]A_{sl}, a_l[3]B_l, a_l[4]B_l, a_l[5]D_l), \end{aligned}$$

where a_k and a_l are S_2-equivariant. By analogy with the previous section we denote

$$\begin{aligned} X_{kl}[1] &= (A_lD_k, A_{sl}E_k, B_lF_k, C_lF_{sk}, A_kD_l, A_{sk}E_l, B_kF_l, C_kF_{sl}) \\ X_{kl}[2] &= (A_{sl}D_k, A_lE_k, B_lF_{sk}, C_lF_k, A_{sk}D_l, A_kE_l, B_kF_{sl}, C_kF_l) \\ X_{kl}[3] &= (B_lD_k, A_{sl}F_{sk}, A_lF_k, C_lE_k, B_kD_l, A_{sk}F_{sl}, A_kF_l, C_kE_l) \\ X_{kl}[4] &= (B_lD_k, A_{sl}F_{sk}, A_lF_k, C_lE_k, B_kD_l, A_{sk}F_{sl}, A_kF_l, C_kE_l) \\ X_{kl}[5] &= (A_lA_k, A_{sl}A_{sk}, B_lB_k, C_lC_k, D_lD_k, E_lE_k, F_lF_k, F_{sl}F_{sk}). \end{aligned}$$

Denote also

$$\begin{aligned} a_{kl}[1] &= \frac{1}{\sigma}\left(-\frac{k_1}{r_1}a_k[1]a_l[1], \frac{k_2}{r_2}a_k[1]a_l[2], k_3a_k[1]a_l[3], 0\right) \\ a_{kl}[2] &= \frac{1}{\sigma}\left(-\frac{k_2}{r_2}a_k[2]a_l[2], \frac{k_1}{r_1}a_k[2]a_l[1], k_3a_k[2]a_l[3], 0\right) \\ a_{kl}[3] &= \frac{1}{\sigma}\left(-k_3a_k[3]a_l[3], \frac{k_2}{r_2}a_k[3]a_l[2], \frac{k_1}{r_1}a_k[3]a_l[1], 0\right) \\ a_{kl}[4] &= \left(-k_3a_k[4]a_l[3], \frac{k_2}{r_2}a_k[4]a_l[2], \frac{k_1}{r_1}a_k[4]a_l[1], 0\right) \\ a_{kl}[5] &= (0, 0, 0, 0), \end{aligned}$$

and finally define

$$b_{kl}[j] \;=\; (a_{kl}[j], a_{lk}[j]).$$

Now we have that $d^2\Phi[j](u_k, u_l)$ is a linear combination of the components of $X_{kl}[j]$ as

$$d^2\Phi[j](u_k, u_l) \;=\; b_{kl}[j].X_{kl}[j].$$

A direct calculation shows that the result of this dot product is in range L and then equation (4.26) becomes

$$L(w_{kl}) \;=\; d^2\Phi(u_k, u_l). \tag{4.27}$$

It can also be checked directly that $d^2\Phi(u_k, u_l)$ is S_2-equivariant and by lemma 12, the solution w_{kl} of system (4.27) must satisfy the equivariance

$$w_{skl}(s\xi) \;=\; sw_{kl}(\xi). \tag{4.28}$$

By writing w_{kl} as a polynomial function of $A_k, A_l \ldots, F_k, F_l$ we reduce the problem to a system of linear algebraic equations. Again we checked that X_{kl} is not a convenient coordinate system and we apply the change of coordinates P_2 defined in the previous section. Thus we denote

$$Y_{kl}[j] = P_2.X_{kl}[j].$$

and let

$$\begin{aligned} d^2\Phi[j](u_k, u_l) &= q_{kl}[j].Y_{kl}[j] \\ w_{kl}[j] &= p_{kl}[j].Y_{kl}[j], \end{aligned} \tag{4.29}$$

where

$$q_{kl}[j] = P_2^{-1} b_{kl}[j]$$

and p_{kl} are the coefficients that we want to calculate. Now we have that Y_{kl} satisfies the equivariance

$$\begin{aligned} Y_{skl}[1] &= Y_{kl}[2] \\ Y_{skl}[3,2] &= Y_{kl}[3,3] \\ Y_{skl}[3,6] &= Y_{kl}[3,7] \\ Y_{skl}[4,2] &= Y_{kl}[4,3] \\ Y_{skl}[4,6] &= Y_{kl}[4,7] \\ Y_{skl}[5,2] &= -Y_{kl}[5,8] \\ Y_{skl}[5,4] &= -Y_{kl}[5,6]. \end{aligned} \tag{4.30}$$

The S_2-equivariance of $d^2\Phi(u_k, u_l)$ and w_{kl} imposes the symmetry constraints (4.30) on the coefficients q_{kl} and p_{kl} respectively.

Substituting (4.29) in system (4.27) we get a system of linear algebraic equations as

$$L_{kl} p_{kl} = q_{kl},$$

where q_{kl} is already known and by permuting rows and columns, the matrix L_{kl} may be written as

$$L_{kl} = \begin{pmatrix} L_{kl}[1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{skl}[1] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{kl}[2] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{skl}[2] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_k[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{sk}[4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_k[5] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_k[6] \end{pmatrix},$$

and the matrices along the diagonal are of the form

$$L_{kl}[j] \;=\; \begin{pmatrix} -d & 0 & 0 & 0 & a \\ 0 & -d & 0 & 0 & b \\ 0 & 0 & -d & 1 & c \\ 0 & 0 & R & -d & 0 \\ a & b & c & 0 & 0 \end{pmatrix},$$

where $d = a^2 + b^2 + c^2$ and

- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = \frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = -k_3 + l_3$ if $j = 1$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = \frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = k_3 + l_3$ if $j = 2$.
- $a = -\frac{k_1}{r_1} - \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} - \frac{l_2}{r_2}$ and $c = k_3 - l_3$ if $j = 3$.
- $a = -\frac{k_1}{r_1} - \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} - \frac{l_2}{r_2}$ and $c = -k_3 - l_3$ if $j = 4$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = k_3 + l_3$ if $j = 5$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = -k_3 + l_3$ if $j = 6$.

Letting c_k be the reordering of p_k corresponding to the permutation of columns of L_{kl} we have that

$$\begin{aligned}
c_{kl}[1] &= (p_{kl}[1,7], p_{kl}[2,1], p_{kl}[3,7], p_{kl}[4,7], p_{kl}[5,2]) \\
c_{kl}[2] &= (p_{kl}[1,5], p_{kl}[2,3], p_{kl}[3,2], p_{kl}[4,2], p_{kl}[5,4]) \\
c_{kl}[3] &= (p_{kl}[1,2], p_{kl}[2,2], p_{kl}[3,5], p_{kl}[4,5], p_{kl}[5,7]) \\
c_{kl}[4] &= (p_{kl}[1,4], p_{kl}[2,4], p_{kl}[3,4], p_{kl}[4,4], p_{kl}[5,5]) \\
c_{kl}[5] &= (p_{kl}[1,6], p_{kl}[2,6], p_{kl}[3,1], p_{kl}[4,1], p_{kl}[5,3]) \\
c_{kl}[6] &= (p_{kl}[1,8], p_{kl}[2,8], p_{kl}[3,8], p_{kl}[4,8], p_{kl}[5,1]),
\end{aligned}$$

and we let d_k be the same reordering of q_k. By eliminating the redundant blocks of the form $L_{s_k s_l}[j]$ we are restricted to solve the systems

$$L_{kl}[j]c_{kl}[j] \;=\; d_{kl}[j]$$

for $1 \leq j \leq 6$, which is a reasonably easy calculation with or without computer algebra.

(b) The rest of the job with Maple

Now, Maple has no problem in doing the rest of the job. Recall that what is left is the computation of

$$V_{kl} \;=\; -d^2\Phi(u_k, w_{ll}) - 2d^2\Phi(u_l, w_{kl}),$$

where u_k, u_l are two of the eigenfuntions generating ker L, w_{ll} was computed in section 4.8.1 and w_{kl} is what we have just calculated. Given the values of u_k, u_l, w_{ll} and w_{kl}, that we know at this point, Maple computes V_{kl} according to the formula above. Then we have one more coefficient of the reduced equations by computing

$$a_{N_2} = \frac{1}{2}\langle u_k^*, V_{kl}\rangle,$$

where we recall that u_k^* is one of the eigenfuntions generating ker L^*. Five of the remaining coefficients of the bifurcation equations are obtained by substituting the mode numbers $\underline{k}$ in the formulae above by $\underline{l}$ and $\underline{m}$ respectively. The results are as follows

	Critical point 1	Critical point 2
a_{N_2}	$-\frac{9.676\sigma-0.142}{\sigma^4}\pi^3$	$-\frac{9.689\sigma-0.119}{\sigma^4}\pi^3$
a_{N_3}	$-\frac{7.693\sigma+3.474}{\sigma^2}\pi^3$	$-\frac{5.902\sigma+0.840}{\sigma^2}\pi^3$
b_{N_1}	$-\frac{8.733\sigma+0.692}{\sigma^2}\pi^3$	$-\frac{8.946\sigma+0.573}{\sigma^2}\pi^3$
b_{N_3}	$-\frac{5.094\sigma+0.386}{\sigma^2}\pi^3$	$-\frac{3.713\sigma-1.092}{\sigma^4}\pi^3$
c_{N_1}	$-\frac{6.386\sigma-1.824}{\sigma^4}\pi^3$	$-\frac{5.902\sigma+0.840}{\sigma^2}\pi^3$
c_{N_2}	$-\frac{5.094\sigma+0.386}{\sigma^4}\pi^3$	$-\frac{4.228\sigma+1.848}{\sigma^2}\pi^3$

4.8.3 Calculation of $a_\lambda, b_\lambda, c_\lambda$

In this section we describe how to obtain a_λ. Then the coefficients b_λ and c_λ come immediately by substituting the mode numbers $\underline{k}$ by $\underline{l}$ and $\underline{m}$ respectively. The calculation of a_λ is very simple comparing with the complication of the previous sections. Recall that the formula is

$$a_\lambda = \langle u_k^*, (d\Phi_\lambda).u_k - d^2\Phi(u_k, L^{-1}E\Phi_\lambda)\rangle.$$

By differentiating Φ with respect to λ we get

$$\Phi_\lambda = (0,0,0,v_3,0)^t.$$

Since Φ_λ is a linear operator, its linearization evaluated at u_k is

$$(d\Phi_\lambda).u_k = (0,0,0,u_k[3],0)^t.$$

Note that Φ_λ is odd, which implies that when $u=0$ we have

$$\Phi_\lambda = 0.$$

All together, these results say that

$$\begin{aligned} a_\lambda &= \langle u_k^*, (d\Phi_\lambda).u_k\rangle \\ &= \int_0^\pi\int_0^\pi\int_0^\pi u_k^*[4]u_k[3]\,dx\,dy\,dz. \end{aligned}$$

The result evaluated by Maple is approximately

	Critical point 1	Critical point 2
a_λ	$2.752\pi^3$	$2.790\pi^3$
b_λ	$1.622\pi^3$	$1.585\pi^3$
c_λ	$3.244\pi^3$	$2.790\pi^3$

4.8.4 Calculation of $a_{\rho_1}, a_{\rho_2}, b_{\rho_1}, b_{\rho_2}, c_{\rho_1}, c_{\rho_2}$

In this section we describe how to calculate a_{ρ_1} and the remaining five coefficients are obtained by analogy. Recall that the formula is

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k - d^2\Phi(u_k, L^{-1}E\Phi_{\rho_1})\rangle.$$

Differentiating Φ with respect to the parameter ρ_1 we get

$$\begin{aligned}
\Phi_{\rho_1}[1] &= -\frac{2}{r_1^3}\frac{\partial^2 v_1}{\partial \xi_1^2} + \frac{1}{r_1^2}\frac{\partial p}{\partial \xi_1} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_1}{\partial \xi_1} \\
\Phi_{\rho_1}[2] &= -\frac{2}{r_1^3}\frac{\partial^2 v_2}{\partial \xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_2}{\partial \xi_1} \\
\Phi_{\rho_1}[3] &= -\frac{2}{r_1^3}\frac{\partial^2 v_3}{\partial \xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_3}{\partial \xi_1} \\
\Phi_{\rho_1}[4] &= -\frac{2}{r_1^3}\frac{\partial^2 \Theta}{\partial \xi_1^2} + \frac{1}{r_1^2}v_1\frac{\partial \Theta}{\partial \xi_1} \\
\Phi_{\rho_1}[5] &= -\frac{1}{r_1^2}\frac{\partial v_1}{\partial \xi_1}.
\end{aligned}$$

By linearizing this operator about $u = 0$ and evaluating at u_k we get

$$\begin{aligned}
(d\Phi_{\rho_1}[1]).u_k &= \left(\frac{2}{r_1^3}k_1^2 a_k[1] - \frac{1}{r_1^2}k_1 a_k[5]\right)A_k \\
(d\Phi_{\rho_1}[2]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[2]A_{sk} \\
(d\Phi_{\rho_1}[3]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[3]B_k \\
(d\Phi_{\rho_1}[4]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[4]B_k \\
(d\Phi_{\rho_1}[5]).u_k &= -\frac{1}{r_1^2}k_1 a_k[1]D_k.
\end{aligned}$$

By inspection of the operator Φ_{ρ_1} we see that $u = 0$ then

$$\Phi_{\rho_1} = 0.$$

All together, the results above say that

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k\rangle.$$

The result evaluated by Maple for the six coefficients is approximately

	Critical point 1	Critical point 2
a_{ρ_1}	0	0
a_{ρ_2}	$8.249\pi^3$	$8.666\pi^3$
b_{ρ_1}	$0.842\pi^3$	$0.567\pi^3$
b_{ρ_2}	$4.373\pi^3$	$4.534\pi^3$
c_{ρ_1}	$6.732\pi^3$	$4.333\pi^3$
c_{ρ_2}	0	0

4.8.4 Calculation of $a_{\rho_1}, a_{\rho_2}, b_{\rho_1}, b_{\rho_2}, c_{\rho_1}, c_{\rho_2}$

In this section we describe how to calculate a_{ρ_1} and the remaining five coefficients are obtained by analogy. Recall that the formula is

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k - d^2\Phi(u_k, L^{-1}E\Phi_{\rho_1})\rangle.$$

Differentiating Φ with respect to the parameter ρ_1 we get

$$\begin{aligned}
\Phi_{\rho_1}[1] &= -\frac{2}{r_1^3}\frac{\partial^2 v_1}{\partial\xi_1^2} + \frac{1}{r_1^2}\frac{\partial p}{\partial\xi_1} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_1}{\partial\xi_1} \\
\Phi_{\rho_1}[2] &= -\frac{2}{r_1^3}\frac{\partial^2 v_2}{\partial\xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_2}{\partial\xi_1} \\
\Phi_{\rho_1}[3] &= -\frac{2}{r_1^3}\frac{\partial^2 v_3}{\partial\xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_3}{\partial\xi_1} \\
\Phi_{\rho_1}[4] &= -\frac{2}{r_1^3}\frac{\partial^2 \Theta}{\partial\xi_1^2} + \frac{1}{r_1^2}v_1\frac{\partial \Theta}{\partial\xi_1} \\
\Phi_{\rho_1}[5] &= -\frac{1}{r_1^2}\frac{\partial v_1}{\partial\xi_1}.
\end{aligned}$$

By linearizing this operator about $u = 0$ and evaluating at u_k we get

$$\begin{aligned}
(d\Phi_{\rho_1}[1]).u_k &= \left(\frac{2}{r_1^3}k_1^2 a_k[1] - \frac{1}{r_1^2}k_1 a_k[5]\right)A_k \\
(d\Phi_{\rho_1}[2]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[2]A_{sk} \\
(d\Phi_{\rho_1}[3]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[3]B_k \\
(d\Phi_{\rho_1}[4]).u_k &= \frac{2}{r_1^3}k_1^2 a_k[4]B_k \\
(d\Phi_{\rho_1}[5]).u_k &= -\frac{1}{r_1^2}k_1 a_k[1]D_k.
\end{aligned}$$

By inspection of the operator Φ_{ρ_1} we see that $u = 0$ then

$$\Phi_{\rho_1} = 0.$$

All together, the results above say that

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k\rangle.$$

The result evaluated by Maple for the six coefficients is approximately

	Critical point 1	Critical point 2
a_{ρ_1}	0	0
a_{ρ_2}	$8.249\pi^3$	$8.666\pi^3$
b_{ρ_1}	$0.842\pi^3$	$0.567\pi^3$
b_{ρ_2}	$4.373\pi^3$	$4.534\pi^3$
c_{ρ_1}	$6.732\pi^3$	$4.333\pi^3$
c_{ρ_2}	0	0

4.8.5 Normal Form and Bifurcation Diagrams

Given a set of equations $f(x,\lambda) = 0$, where f is obtained by a Liapunov-Schmidt reduction we can find the topology of the bifurcation diagram, but not the stability. The corresponding differential equation is one of

(a) $\dot{x} + f(x,\lambda) = 0$.

(b) $\dot{x} = f(x,\lambda)$.

In order to get information about stability we must know if our mapping f satisfies (a) or (b). In previous sections a 3rd order truncation of f was calculated for the (0,1,1)-(1,1,1)-(2,0,1) mode interaction on the Bénard problem. The result is a $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant as

$$\begin{aligned} f_1(x,y,z,\lambda,\rho_1,\rho_2) &= (a_{N_1}N_1 + a_{N_2}N_2 + a_{N_3}N_3 + a_\lambda\lambda + a_{\rho_1}\rho_1 + a_{\rho_2}\rho_2)x \\ f_2(x,y,z,\lambda,\rho_1,\rho_2) &= (b_{N_1}N_1 + b_{N_2}N_2 + b_{N_3}N_3 + b_\lambda\lambda + b_{\rho_1}\rho_1 + b_{\rho_2}\rho_2)y \\ f_3(x,y,z,\lambda,\rho_1,\rho_2) &= (c_{N_1}N_1 + c_{N_2}N_2 + c_{N_3}N_3 + c_\lambda\lambda + c_{\rho_1}\rho_1 + c_{\rho_2}\rho_2)z \end{aligned}$$

where $N_1 = x^2$, $N_2 = y^2$, $N_3 = z^2$. The bifurcation parameter λ is the Rayleigh number with the origin shifted and ρ_1, ρ_2 are the unfolding parameters that have to do with the lengths of the domain. The coefficients a, b, c are computed in previous sections by a Liapunov-Schmidt reduction.

When $\rho_1 = \rho_2 = 0$, the linearization of f about the origin is

$$df = \begin{pmatrix} a_\lambda\lambda & 0 & 0 \\ 0 & b_\lambda\lambda & 0 \\ 0 & 0 & c_\lambda\lambda \end{pmatrix},$$

which has three negative real eigenvalues if $\lambda < 0$ since $a_\lambda, b_\lambda, c_\lambda$ are positive numbers as in section 4.8.3. Thus an equation in the form (a) for this particular f leads to instability of the solution $(x,y,z) = (0,0,0)$. This means instability of the translation-invariant solution $u = 0$ to the Bénard problem when the Rayleigh number is below the critical value, which does not make sense physically. In the same way, we see that an equation in the form (b) gives the correct stability.

In appendix A we analyse a normal form with $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry. The stabilities are given under the assumption that the mapping f satisfies the differential equation (a). In order to apply the results obtained to this particular problem it is enough to change the signs of all coefficients in f and reduce the result to the equivalent normal form.

Some of the coefficients depend on the Prandtl number σ. From now on we assume that the fluid inside the box is water, which corresponds to

$$\sigma \approx 7.03.$$

Substituting this number in the coefficients calculated in previous sections and reversing sign we get

	Critical point 1	Critical point 2
a_{N_1}	11.962	12.421
a_{N_2}	42.589	42.657
a_{N_3}	36.111	26.371
a_λ	−85.328	−86.515
a_{ρ_1}	0	0
a_{ρ_2}	−255.76	−268.71
b_{N_1}	38.951	39.893
b_{N_2}	6.6793	6.3503
b_{N_3}	22.711	15.692
b_λ	−50.294	−49.144
b_{ρ_1}	−26.093	−17.573
b_{ρ_2}	−135.59	−140.58
c_{N_1}	26.935	26.371
c_{N_2}	22.711	19.799
c_{N_3}	18.297	12.421
c_λ	−100.59	−86.515
c_{ρ_1}	−208.75	−134.35
c_{ρ_2}	0	0

In order to compute the normal form for f we apply a $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalence as

$$H(x,y,z,\lambda,\rho_1,\rho_2) = Sf(X,\Lambda,\rho_1,\rho_2) \tag{4.31}$$

where

$$X(x,y,z,\lambda,\rho_1,\rho_2) = (Ax, By, Cz)$$
$$\Lambda(\lambda,\rho_1,\rho_2) = \lambda - \frac{a_{\rho_1}}{a_\lambda}\rho_1 - \frac{a_{\rho_2}}{a_\lambda}\rho_2$$
$$S(x,y,z,\lambda,\rho_1,\rho_2) = \begin{pmatrix} D & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix}$$

and A, B, C, D, E, F are positive constants. Substituting X, Λ and S in (4.31) leads to

$$\begin{aligned}
H_1(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (a_{N_1}A^3Dx^2 + a_{N_2}AB^2Dy^2 + a_{N_3}AC^2Dz^2 + a_\lambda AD\lambda)x \\
H_2(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (b_{N_1}A^2BEx^2 + b_{N_2}B^3Ey^2 + b_{N_3}BC^2Ez^2 + b_\lambda BE\lambda + \tilde{r}_1)y \\
H_3(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (c_{N_1}A^2CFx^2 + c_{N_2}B^2CFy^2 + c_{N_3}C^3Fz^2 + c_\lambda CF\lambda + \tilde{r}_2)z,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{r}_1 &= BE\left(b_{\rho_1} - \frac{a_{\rho_1}b_\lambda}{a_\lambda}\right)\rho_1 + BE\left(b_{\rho_2} - \frac{a_{\rho_2}b_\lambda}{a_\lambda}\right)\rho_2 \\
\tilde{r}_2 &= CF\left(c_{\rho_1} - \frac{a_{\rho_1}c_\lambda}{a_\lambda}\right)\rho_1 + CF\left(c_{\rho_2} - \frac{a_{\rho_2}c_\lambda}{a_\lambda}\right)\rho_2.
\end{aligned}$$

By imposing the conditions

$$
\begin{aligned}
|a_{N_1}|A^3D &= 1\\
|b_{N_2}|B^3E &= 1\\
|c_{N_3}|C^3F &= 1\\
|a_\lambda|AD &= 1\\
|b_\lambda|BE &= 1\\
|c_\lambda|CF &= 1
\end{aligned}
$$

we get H in the desired form

$$
\begin{aligned}
H_1(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (\epsilon_1 x^2 + n_1 y^2 + n_2 z^2 + \epsilon_2\lambda)x\\
H_2(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (n_3 x^2 + \epsilon_3 y^2 + n_4 z^2 + \epsilon_4\lambda + \tilde{r}_1)y\\
H_3(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (n_5 x^2 + n_6 y^2 + \epsilon_5 z^2 + \epsilon_6\lambda + \tilde{r}_2)z,
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_1 &= \operatorname{sgn}(a_{N_1})\\
\epsilon_2 &= \operatorname{sgn}(a_\lambda)\\
\epsilon_3 &= \operatorname{sgn}(b_{N_2})\\
\epsilon_4 &= \operatorname{sgn}(b_\lambda)\\
\epsilon_5 &= \operatorname{sgn}(c_{N_3})\\
\epsilon_6 &= \operatorname{sgn}(c_\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
n_1 &= \left|\frac{b_\lambda}{b_{N_2}a_\lambda}\right| a_{N_2}\\
n_2 &= \left|\frac{c_\lambda}{c_{N_3}a_\lambda}\right| a_{N_3}\\
n_3 &= \left|\frac{a_\lambda}{a_{N_1}b_\lambda}\right| b_{N_1}\\
n_4 &= \left|\frac{c_\lambda}{c_{N_3}b_\lambda}\right| b_{N_3}\\
n_5 &= \left|\frac{a_\lambda}{a_{N_1}c_\lambda}\right| c_{N_1}\\
n_6 &= \left|\frac{b_\lambda}{b_{N_2}c_\lambda}\right| c_{N_2}.
\end{aligned}
$$

The unfolding parameters are

$$
\begin{aligned}
\tilde{r}_1 &= \frac{1}{|b_\lambda|}\left(b_{\rho_1} - \frac{a_{\rho_1}b_\lambda}{a_\lambda}\right)\rho_1 + \frac{1}{|b_\lambda|}\left(b_{\rho_2} - \frac{a_{\rho_2}b_\lambda}{a_\lambda}\right)\rho_2\\
\tilde{r}_2 &= \frac{1}{|c_\lambda|}\left(c_{\rho_1} - \frac{a_{\rho_1}c_\lambda}{a_\lambda}\right)\rho_1 + \frac{1}{|c_\lambda|}\left(c_{\rho_2} - \frac{a_{\rho_2}c_\lambda}{a_\lambda}\right)\rho_2.
\end{aligned}
$$

By substitiing the derivatives of a, b, c given above in the formulae for ϵ_j and n_j we get

$$\begin{aligned}\epsilon_1 = \epsilon_3 = \epsilon_5 &= 1\\ \epsilon_2 = \epsilon_4 = \epsilon_6 &= -1\end{aligned}$$

and

	Critical point 1	Critical point 2
n_1	3.758	3.816
n_2	2.327	2.123
n_3	5.524	5.654
n_4	2.482	2.224
n_5	1.910	2.123
n_6	1.700	1.771

Now the unfolding parameters are approximately

	Critical point 1	Critical point 2
$\bar{r}_1$	$-0.519\rho_1 + 0.302\rho_2$	$-0.358\rho_1 + 0.245\rho_2$
$\bar{r}_2$	$-2.075\rho_1 + 2.997\rho_2$	$-1.553\rho_1 + 3.106\rho_2$

Taking these numbers to appendix A, the bifurcation diagrams can be drawn directly by reading the tables. The nondegeneracy conditions are satisfied, since by using the notation in the appendix we have

	Critical point 1	Critical point 2
(a)	−2.758	−2.816
(b)	−1.327	−1.123
(c)	−4.524	−4.654
(d)	−1.482	−1.224
(e)	−0.910	−1.123
(f)	−0.700	−0.771
(g)	−19.76	−20.57
(h)	−3.444	−3.508
(i)	−3.220	−2.939
(j)	−6.802	−6.354
(k)	6.144	5.341
(l)	3.980	3.369
(m)	11.24	10.26

and they are all nonzero. In appendix A we use the signs of these constants to get all the information about existence and criticality of bifurcating branches. Since the signs are the same for the two critical points they lead to the same bifurcation diagrams. From now on we concentrate on point 1. Before drawing the bifurcation diagrams we make a convention about the labels of possible branches.

(0) Translation-invariant solution $u = 0$.

(1) Pure mode (0,1,1).

(2) Pure mode (1,1,1).

(3) Pure mode (2,0,1).

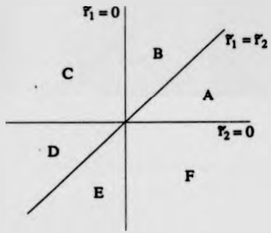
(4) Mixed mode (0,1,1)-(1,1,1).

(5) Mixed mode (0,1,1)-(2,0,1).

(6) Mixed mode (1,1,1)-(2,0,1).

(7) Mixed mode (0,1,1)-(1,1,1)-(2,0,1).

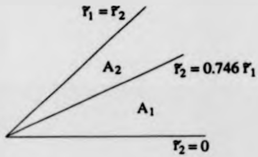
Now the unfolding parameter space is divided into six distinct regions according to the order of primary bifurcations when the parameter is increased. This division is the following

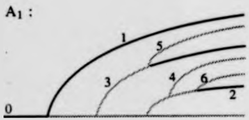


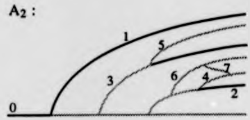
Then some of these regions need a subdivision according to the order of secondary bifurcations. This will be illustrated when appropriate and we proceed by drawing the bifurcation diagrams.

Region A

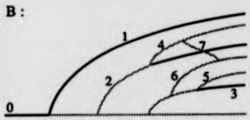
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.





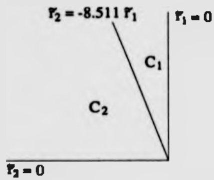


Region B

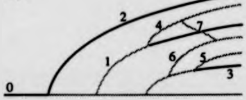


Region C

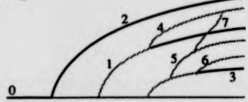
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



C_1 :

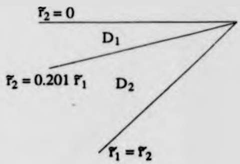


C_2 :

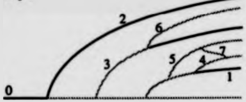


Region D

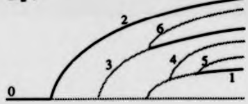
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



D_1 :

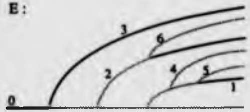


D_2 :

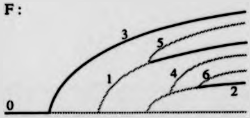


Region E

E :



Region F



4.9 Interaction of Two Modes in a Box with Square Cross Section

Now we assume that $r = r_c$ is such that the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ bifurcate simultaneously when the Rayleigh number is increased across a critical value $R = R_c$. By imposing the extra condition $r_1 = r_2$ we have that ker L is either 2, 3 or 4-dimensional depending on the mode numbers as

1. $V_k \oplus V_l$ if $k_1 = k_2$ and $l_1 = l_2$
2. $V_k \oplus V_l \oplus V_{sl}$ if $k_1 = k_2$ and $l_1 \neq l_2$
3. $V_k \oplus V_{sk} \oplus V_l \oplus V_{sl}$ if $k_1 \neq k_2$ and $l_1 \neq l_2$

and ker L^* is

1. $V_k^* \oplus V_l^*$ if $k_1 = k_2$ and $l_1 = l_2$
2. $V_k^* \oplus V_l^* \oplus V_{sl}^*$ if $k_1 = k_2$ and $l_1 \neq l_2$
3. $V_k^* \oplus V_{sk}^* \oplus V_l^* \oplus V_{sl}^*$ if $k_1 \neq k_2$ and $l_1 \neq l_2$

where V_k and V_k^* are defined in section 4.7.1 for any given set of mode numbers $\underline{k}$. For the particular case of the Bénard convection we did not find any mode interaction of type 1. We performed a Liapunov-Schmidt reduction for a mode interaction of type 2 which gives a 3-dimensional system of ODEs. For the particular modes chosen, a 3rd order truncation is enough to determine the bifurcation diagrams and the analysis described in section 4.8 applies directly. Further analysis of the points of type 3 is left for further work. The reason for not doing it here is that an expansion of the reduced equations up to 3rd order would not be enough to catch all the features of the bifurcation diagrams and we are not technically prepared to go up to higher order yet.

Now we consider a bifurcation of type 2 with mode numbers

$$\begin{aligned} \underline{k} &= (1,1,1) \\ \underline{l} &= (0,1,1). \end{aligned}$$

A calculation shows that these modes bifurcate simultaneously when the unfolding parameters are

$$r_{1c} = r_{2c} \approx 1.687$$

and the bifurcation parameter is

$$R_c \approx 7.024.$$

As in section 4.5.2, the bifurcation equations obtained by a Liapunov-Schmidt reduction have $\mathbf{Z}_2 \oplus \mathbf{D}_4$-symmetry. Thus, up to 3rd order they are of the form

$$\begin{aligned}
f_1 &= [a_{N_1} N_1 + a_{N_2} N_2 + a_\lambda \lambda + a_{\rho_1+\rho_2}(\rho_1 + \rho_2)]\, x \\
f_2 &= [b_{N_1} N_1 + b_{N_2} N_2 + c\delta + b_\lambda \lambda + b_{\rho_1+\rho_2}(\rho_1 + \rho_2) + b_{\rho_1-\rho_2}(\rho_1 - \rho_2)]\, y_1 \\
f_3 &= [b_{N_1} N_1 + b_{N_2} N_2 - c\delta + b_\lambda \lambda + b_{\rho_1+\rho_2}(\rho_1 + \rho_2) - b_{\rho_1-\rho_2}(\rho_1 - \rho_2)]\, y_2
\end{aligned}$$

where

$$N_1 = x^2 \qquad N_2 = y_1^2 + y_2^2 \qquad \delta = y_2^2 - y_1^2$$

and

$$\lambda = R - R_c \qquad \rho_1 = r_1 - r_{1c} \qquad \rho_2 = r_2 - r_{2c}.$$

The coefficients in the formulae for f are computed as in section 4.5.2 and the result for $\sigma = 7.03$ (Prandtl number for the water) is

$$\begin{aligned}
a_{N_1} &\approx 10.715 \\
a_{N_2} &\approx 35.738 \\
a_\lambda &\approx -59.780 \\
a_{\rho_1+\rho_2} &\approx -55.835 \\
b_{N_1} &\approx 46.241 \\
b_{N_2} &\approx 15.643 \\
b_\lambda &\approx -80.288 \\
b_{\rho_1+\rho_2} &\approx -96.732 \\
b_{\rho_1-\rho_2} &\approx 96.732 \\
c &\approx 5.6526
\end{aligned}$$

In order to compute the normal form and bifurcation diagrams we apply a $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivariant transformation

$$H(x, y_1, y_2, \lambda, \rho_1 + \rho_2, \rho_1 - \rho_2) = Sf(X, \Lambda, \rho_1 + \rho_2, \rho_1 - \rho_2) \tag{4.32}$$

where

$$X(x, y_1, y_2, \lambda, \rho_1 + \rho_2, \rho_1 - \rho_2) = (Ax, By_1, By_2)$$
$$\Lambda(\lambda) = \lambda - \frac{a_{\rho_1+\rho_2}}{a_\lambda}(\rho_1 + \rho_2)$$
$$S(x, y_1, y_2, \lambda, \rho_1 + \rho_2, \rho_1 - \rho_2) = \begin{pmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}$$

and A, B, C, D are positive constants. By substituting X, Λ and S in (4.32) we get

$$\begin{aligned} H_1(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= [a_{N_1} A^3 C x^3 + a_{N_2} AB^2 C(y_1^2 + y_2^2) + a_\lambda AC\lambda]x \\ H_2(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= [b_{N_1} A^2 BD x^2 + b_{N_2} B^3 D(y_1^2 + y_2^2) + cB^3 D(y_2^2 - y_1^2) \\ &\quad + b_\lambda BD\lambda + \bar\rho_1 + \bar\rho_2]y_1 \\ H_3(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= [b_{N_1} A^2 BD x^2 + b_{N_2} B^3 D(y_1^2 + y_2^2) - cB^3 D(y_2^2 - y_1^2) \\ &\quad + b_\lambda BD\lambda + \bar\rho_1 - \bar\rho_2]y_2. \end{aligned}$$

where

$$\begin{aligned} \bar\rho_1 &= BD\left(b_{\rho_1+\rho_2} - \frac{a_{\rho_1+\rho_2} b_\lambda}{a_\lambda}\right)(\rho_1 + \rho_2) \\ \bar\rho_2 &= BDb_{\rho_1-\rho_2}(\rho_1 - \rho_2). \end{aligned}$$

By imposing the conditions

$$\begin{aligned} |a_{N_1}|A^3 C &= 1 \\ |a_\lambda|AC &= 1 \\ |c|B^3 D &= 1 \\ |b_\lambda|BD &= 1 \end{aligned}$$

we get H in the desired form

$$\begin{aligned} H_1(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= \left[\epsilon_1 x^2 + n_1(y_1^2 + y_2^2) + \epsilon_2\lambda\right] x \\ H_2(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= \left[n_2 x^2 + n_3(y_1^2 + y_2^2) + \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda + \bar\rho_1 + \bar\rho_2\right] y_1 \\ H_3(x, y_1, y_2, \lambda, \bar\rho_1, \bar\rho_2) &= \left[n_2 x^2 + n_3(y_1^2 + y_2^2) - \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda + \bar\rho_1 - \bar\rho_2\right] y_2 \end{aligned}$$

where

$$\begin{aligned} \epsilon_1 &= \operatorname{sgn}(a_{N_1}) \\ \epsilon_2 &= \operatorname{sgn}(a_\lambda) \\ \epsilon_3 &= \operatorname{sgn}(c) \\ \epsilon_4 &= \operatorname{sgn}(b_\lambda) \end{aligned}$$

and

$$\begin{aligned} n_1 &= \left|\frac{b_\lambda}{c a_\lambda}\right| a_{N_1} \\ n_2 &= \left|\frac{a_\lambda}{a_{N_1} b_\lambda}\right| b_{N_1} \\ n_3 &= \left|\frac{1}{c}\right| b_{N_2}. \end{aligned}$$

The unfolding parameters are

$$\begin{aligned} \bar{\rho}_1 &= \frac{1}{|b_\lambda|}\left(b_{\rho_1+\rho_2} - \frac{a_{\rho_1+\rho_2} b_\lambda}{a_\lambda}\right)(\rho_1+\rho_2) \\ \bar{\rho}_2 &= \frac{1}{|b_\lambda|} b_{\rho_1-\rho_2}(\rho_1-\rho_2). \end{aligned}$$

By substituting the derivatives of a, b, c given above in the formulae for ϵ_j and n_j we get

$$\begin{aligned} \epsilon_1 = \epsilon_3 &= 1 \\ \epsilon_2 = \epsilon_4 &= -1 \end{aligned}$$

and

$$\begin{aligned} n_1 &\approx 8.491 \\ n_2 &\approx 3.213 \\ n_3 &\approx 2.767. \end{aligned}$$

Now the unfolding parameters are

$$\begin{aligned} \bar{\rho}_1 &\approx 0.663(\rho_1+\rho_2) \\ \bar{\rho}_2 &\approx 1.205(\rho_1-\rho_2). \end{aligned}$$

A complete stability analysis of steady solutions of the system

$$\begin{aligned} \dot{x} &= H_1(x, y_1, y_2, \lambda, \bar{\rho}_1, \bar{\rho}_2) \\ \dot{y}_1 &= H_2(x, y_1, y_2, \lambda, \bar{\rho}_1, \bar{\rho}_2) \\ \dot{y}_2 &= H_3(x, y_1, y_2, \lambda, \bar{\rho}_1, \bar{\rho}_2) \end{aligned}$$

is performed in appendix B. The results are condensed into tables giving the branches of solutions and their criticality with respect to the bifurcation parameter λ if the system is nondegenerate. Our particular mapping H satisfies the nondegeneracy conditions since we have

$$\begin{aligned} (a) &\approx -2.213 \\ (b) &\approx 2.767 \\ (c) &\approx 1.767 \\ (d) &\approx -5.724 \\ (e) &\approx -6.724 \\ (f) &\approx -24.52 \\ (g) &\approx -26.28 \end{aligned}$$

and they are all nonzero. We proceed by drawing the bifurcation diagrams for the case when the mapping H is $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivariant. This happens when $\tilde{\rho}_2 = 0$. Then we show the effects of breaking the $\mathbf{S}_2$ symmetry with the parameter $\tilde{\rho}_2$. Before drawing the bifurcation diagrams we make a convention about the labels of possible branches.

(0) Translation-invariant solution $u = 0$.

(1) Pure mode (1,1,1).

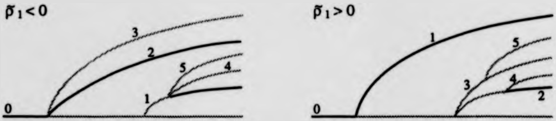
(2) Pure modes (0,1,1) and (1,0,1).

(3) Mixed mode (0,1,1)-(1,0,1).

(4) Mixed modes (1,1,1)-(0,1,1) and (1,1,1)-(1,0,1).

(5) Mixed mode (1,1,1)-(0,1,1)-(1,0,1).

Now the bifurcation diagrams depend on the sign of $\tilde{\rho}_1$ as



We proceed by breaking the $\mathbf{S}_2$ symmetry with the parameter $\tilde{\rho}_2$. Now the modes (0,1,1) and (1,0,1) are no longer conjugate. In order to make the bifurcation diagrams clear we need new labels

(2.1) Pure mode (0,1,1).

(2.2) Pure mode (1,0,1).

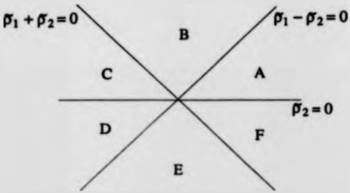
(4.1) Mixed mode (1,1,1)-(0,1,1).

(4.2) Mixed mode (1,1,1)-(1,0,1).

(6) Mixed mode (1,0,1)-(0,1,1).

(7) Mixed mode (1,1,1)-(0,1,1)-(1,0,1).

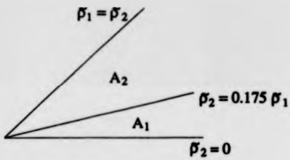
The unfolding parameter space is divided into six distinct regions according to the order of primary bifurcations when the bifurcation parameter is increased. This division is the following

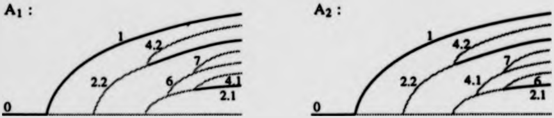


Then some of these regions need a subdivision according to the order of secondary bifurcations. This will be illustrated when appropriate and we proceed by drawing the bifurcation diagrams.

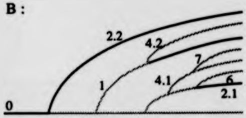
Region A

This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.

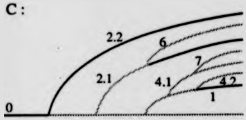




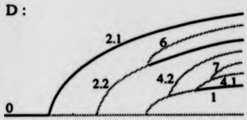
Region B



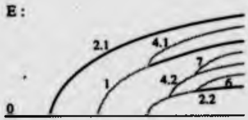
Region C



Region D

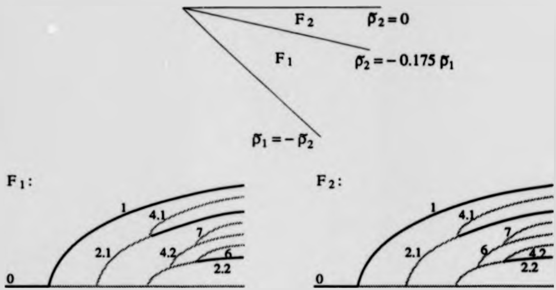


Region E



Region F

This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



Part II

Bistable Chaos: Bifurcation Analysis

Chapter 5

Introduction

5.1 Introduction

In this part, we describe a different example of bifurcation in symmetric systems, namely the bifurcation phenomena arising in the circuit equations modelling the *chaotic Van der Pol-Duffing oscillator* shown in figure 5.1. These results have been published in Gomes and King [32]. The experimental work done by Gomes [33] is much less complete than what we present here and it was a motivation for this more detailed investigation. The bifurcations and dynamics in this oscillator are due to the presence of the nonlinear negative resistor N whose $I - V$ characteristic we model with the polynomial

$$I_N(V) = \nu + aV + bV^3 \qquad \text{where} \qquad a < 0 \text{ and } b > 0.$$

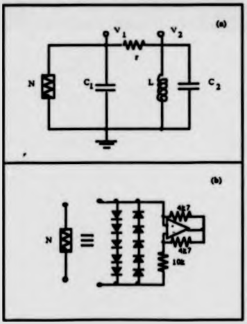


Figure 5.1: Circuit diagram for a chaotic Van der Pol-Duffing oscillator.

Previous investigations of this circuit and its variants have reported the main bifurcation sequence shown in figure 5.2(a) [27, 28, 37, 31]. The main sequence begins with a symmetry-breaking transition (via a pitchfork bifurcation) from S^0 to the *conjugate* attractors S^+ and S^-. Both attractors undergo a Hopf bifurcation followed by a sequence of period doubling bifurcations to chaos. The bifurcations from the S^+ and S^- branches occur at the *same* parameter values. This region of bistability ends with a *symmetry-increasing* bifurcation [24, 38] (via a crisis) to a single chaotic attractor C^0. A one parameter family of the described bifurcation sequences is expected to contain an element for which the pitchfork and Hopf bifurcations occur at the same parameter value. This bifurcation has generically codimension 2 and it has first been described by Takens [41, 42] and Bogdanov [20]. It is known as a Takens-Bogdanov (TB) bifurcation.

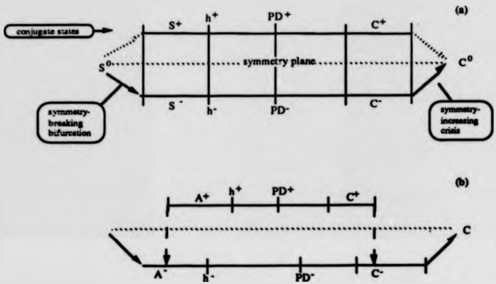


Figure 5.2: Main bifurcation sequences: (a) Symmetric bifurcation sequence; (b) Asymmetric bifurcation sequence.

This Z_2-symmetric sequence is non-generic. It is non-generic because the pitchfork bifurcation from S^0 to $S^{\pm}$ is non-generic. The generic picture for the pitchfork is given by the elementary cusp catastrophe. We show that the presence of the cusp catastrophe in the steady state problem gives rise to the bifurcation sequence shown in figure 5.2(b).

In section 5.2 we describe three nonlinear circuits obtained as limiting cases of figure 5.1, and the TB singularity since it plays an essential role in later chapters. We also show that the equations for one of the circuits can be transformed into the model used by Boissonade and De Kepper [21] to interpret bistability and oscillations for chemical reactions in continuously stirred tank reactors.

In chapters 6 and 7, we give our analytic and numerical results for the chaotic Van der Pol oscillator. In chapter 6 results for the symmetric system ($\nu = 0$) are given. We find a TB line of codimension 2 in the parameter space. This line contains a degenerate point of codimension 3 where a change of stability occurs. As described in Guckenheimer and Holmes [35], several codimension 1 bifurcations (pitchfork, Hopf, homoclinic, saddle-node of limit cycles when $\mathbf{Z}_2$-symmetry is assumed) meet at a codimension 2 degeneracy of the TB type. Homoclinic bifurcations and saddle-node of limit cycles are global bifurcations and as such they require numerical methods to be tracked away from a small neighbourhood of the codimension 2 point. In our numerical experiments the value of one of the parameters has been fixed so that we are left with a 2-dimensional parameter space that intersects transversally the TB line. In chapter 7, we use numerical simulations to obtain a bifurcation set for the asymmetric system. Again one of the parameters is fixed (the same as in section 6). Now we are left with a 3-dimensional parameter space that intersects transversally a TB cusped sheet. The bifurcation set will be described by restricting to several 2-dimensional sections.

5.2 Circuits and Singularities

In this section we describe three limiting cases of circuit 5.1. By analysing the model equations we find singularities that are also present in the chaotic Van der Pol-Duffing oscillator. There is a considerable advantage to this approach. By taking the limits we get simpler systems. Some of the complicated behaviour of circuit 5.1 is no longer present, which makes easier the concentration on singularities. All together, these limits give a good intuition for what is going to happen in the full system and introduce some machinery in an organized way.

5.2.1 Limiting Cases

The three nonlinear circuits shown in figure 5.3 are obtained as limiting cases of figure 5.1. The circuit equation for figure 5.3(a) is

$$\dot{V}_1 = -\frac{1}{C_1}\left(bV_1^3 + aV_1 + \nu\right)$$

which may also be written as

$$\dot{V}_1 = -\frac{1}{C_1}\frac{dF}{dV_1}$$

where F is the potential

$$F(V_1) = \frac{1}{4}bV_1^4 + \frac{1}{2}aV_1^2 + \nu V_1.$$

The steady states of this circuit are completely characterized by the canonical cusp catastrophe.

We recall some results from elementary catastrophe theory [40]. Given the potential, F, the *equilibrium manifold M* is defined by the equation

$$\nabla_x F = 0,$$

where the subscript x indicates that the gradient is with respect to the state variables. In the present case this yields for M the equation

$$bV_1^3 + aV_1 + \nu = 0.$$

By varying the *control* parameters a and ν the familiar folded surface of the canonical cusp catastrophe manifold (see Poston and Stewart [39]) may be generated. The folds of this surface when projected onto the (a, ν) plane yield the cusp

$$b\left(\frac{\nu}{2}\right)^2 = \left(\frac{-a}{3}\right)^3.$$

Inside the cusp the potential F has two minima whose relative depth is determined by ν. In this region the system is said to be *bistable*.

The circuit in figure 5.3(b) is a Van der Pol oscillator and may be obtained from figure 5.1 either by setting $C_1 = 0$ or $r = 0$. Assuming $r = 0$ yields Van der Pol's equation

$$\begin{aligned} \dot{V}_1 &= -\frac{1}{C_1}\left(bV_1^3 + aV_1 + \nu + I_L\right) \\ \dot{I}_L &= \frac{1}{L}V_1 \end{aligned}$$

which in more familiar form is

$$\ddot{V}_1 + \frac{1}{C_1}\left(a + 3bV_1^2\right)\dot{V}_1 + \frac{1}{LC_1}V_1 = 0.$$

The bistability present in the previous circuit has disappeared (but see Zeeman [43] for a catastrophe theory interpretation).

The circuit in figure 5.3(c) is a Van der Pol-Duffing oscillator. The circuit equations are

$$\begin{aligned} \dot{V}_1 &= -\frac{1}{C_1}\left(bV_1^3 + aV_1 + I_L + \nu\right) \\ \dot{I}_L &= \frac{V_1}{L} - \frac{r}{L}I_L. \end{aligned} \tag{5.1}$$

The reader will note that there is little difference between the circuits of figures 5.3(b) and (c). Nevertheless the introduction of the resistor r has a profound effect: Not only is bistability recovered but, as will be shown below, we also have a cusped curve of TB codimension 2 bifurcations (TB cusp).

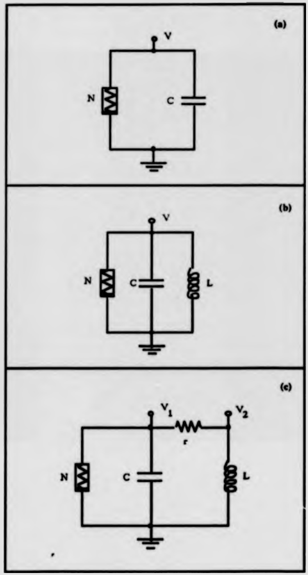


Figure 5.3: Three nonlinear circuits embedded in the chaotic Van der Pol-Duffing circuit: (a) nonlinear RC circuit; (b) Van der Pol oscillator; (c) Van der Pol-Duffing oscillator.

5.2.2 A Takens-Bogdanov Cusp

Before discussing system (5.1) further, we bring it into dimensionless form with the scaling

$$X = \sqrt{\frac{bL}{rC_1}}V_1 \qquad Z = \sqrt{\frac{bLr}{C_1}}I_L \qquad \tau = \frac{r}{L}t$$

which yields the system

$$\begin{aligned} \dot{X} &= -(X^3 - \alpha X + \mu) - \Gamma Z \\ \dot{Z} &= X - Z, \end{aligned} \tag{5.2}$$

where differentiation is with respect to τ and the three parameters are $\alpha = \frac{L}{rC_1}a$, $\mu = \sqrt{\frac{bL^3}{r^3C_1^3}}\nu$, and $\Gamma = \frac{L}{r^2C_1}$.

For ease of interpretation we write system (5.2) as the second order equation

$$\ddot{X} + \left(-(\alpha - 1) + 3X^2\right)\dot{X} + \frac{\partial F}{\partial X} = 0 \tag{5.3}$$

where F is the potential

$$F(X) = \frac{1}{4}X^4 - \frac{1}{2}(\alpha - \Gamma)X^2 + \mu X. \tag{5.4}$$

When $\mu = 0$ the potential F has one minima for $\alpha < \Gamma$ and two minima of equal depth for $\alpha > \Gamma$. In the physics literature the point $\alpha = \Gamma$ is called a 'second order phase transition' point. Examination of the friction term in equation (5.3) shows that there is another 'phase transition' leading to the onset of oscillations with critical point $\alpha = 1$. The two critical points coincide at $(\alpha, \Gamma) = (1, 1)$. In the bifurcation theory literature this critical point is often referred to as the 'nilpotent linear part' or the 'Takens-Bogdanov codimension 2 singularity' (see figure 5.4 and chapter 7 of Guckenheimer and Holmes [35]).

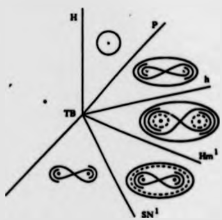


Figure 5.4: Bifurcation set unfolding a Takens-Bogdanov point with $\mathbf{Z}_2$-symmetry. Crosses are saddle points, black (grey) dots are stable (unstable) steady states and solid (dashed) loops are stable (unstable) limit cycles.

When $\mu \neq 0$ we have, in regards to the steady state problem, the same situation as for our first nonlinear circuit above. Now the equilibrium manifold is given by

$$X^3 - (\alpha - \Gamma)X + \mu \; = \; 0 \tag{5.5}$$

and the region where F has a double well is inside the cusped sheet

$$\left(\frac{\mu}{2}\right)^2 \; = \; \left(\frac{\alpha - \Gamma}{3}\right)^3. \tag{5.6}$$

In addition to steady states we also have oscillations. The coincidence of steady state and Hopf bifurcations now occur to yield a *TB-cusp*.

5.2.3 A Cross-Shaped Diagram

A complete picture of the bifurcation set of system (5.1) is given by showing that this system is equivalent to a class of models of chemical reactors described by Guckenheimer [34] and used by him to correct some features in Boissonade and De Kepper's 'cross-shaped diagram' [21]. The cross-shaped diagram was used by Boissonade and De Kepper to summarize some general relationships between bistability and oscillations for chemical reactions in continuously stirred tank reactors. They introduced model (5.2), where X and Y denote concentrations of chemical species, as a schematic representation of circumstances where a reactor which has two stable steady states can become oscillatory.

For ease of exposition we proceed by giving a summary of Guckenheimer's analysis. System (5.2) commutes with the action of the group $\mathbf{Z}_2$

$$(X, Z) \;\mapsto\; (-X, -Z)$$

if and only if $\mu = 0$. In this symmetric case the steady states are

$$(X, Z) \;=\; \begin{cases} (0,0) & \text{if } \alpha \leq \Gamma \\ (0,0) \text{ and } (\pm\sqrt{\alpha - \Gamma}, \pm\sqrt{\alpha - \Gamma}) & \text{if } \alpha > \Gamma. \end{cases}$$

Thus there is a pitchfork bifurcation when α is increased past Γ. The characteristic polynomial of the linearization about the origin is $\lambda^2 + (1 - \alpha)\lambda + (\Gamma - \alpha)$. Thus $(0,0)$ undergoes a Hopf bifurcation when $\alpha = 1$, $\Gamma > \alpha$. There is a TB singularity at $(\alpha, \Gamma) = (1, 1)$. The linearization about $(\pm\sqrt{\alpha - \Gamma}, \pm\sqrt{\alpha - \Gamma})$ has characteristic polynomial $\lambda^2 + (2\alpha - 3\Gamma + 1)\lambda + 2(\alpha - \Gamma)$. Thus the nontrivial steady states undergo a Hopf bifurcation when $2\alpha - 3\Gamma + 1 = 0$, $\alpha \geq \Gamma$. Nonlinear analysis is required for the stability calculations. These are performed in Guckenheimer's paper and the resulting bifurcation set for $\mu = 0$ is shown in figure 5.4.

To complete our picture we need to see how the bifurcation set in figure 5.4 evolves as we vary μ. It can be seen that steady state bifurcations occur by crossing the cusped sheet (5.6) in the (α, Γ, μ)-space. This bifurcation is a pitchfork if $\mu = 0$ and a fold if $\mu \neq 0$. At a fold point the state is $\left(\mathrm{sgn}(\mu)\sqrt{\frac{\alpha-\Gamma}{3}}, \mathrm{sgn}(\mu)\sqrt{\frac{\alpha-\Gamma}{3}}\right)$. By linearizing about this point we find a TB bifurcation at $\Gamma = 1$. On a section of constant $\alpha > 1$ the bifurcation set is the 'cross-shaped diagram' given in Guckenheimer [34] and reproduced here in figure 5.5.

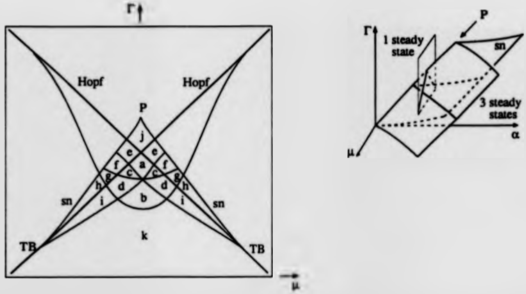


Figure 5.5: Inside the box: Guckenheimer's cross-shaped diagram for system (5.2). Lower case letters correspond to phase portraits from figure 7.3. Only regions of bistability are labeled. Other labels represent bifurcations listed in table 6.1. Outside the box: cusped-sheet dividing the parameter space according to the number of steady states and section where the cross-shaped diagram is found.

Chapter 6

The Symmetric System

6.1 Introduction

This chapter is concerned with the bifurcation analysis of the chaotic Van der Pol-Duffing oscillator when the characteristic of the negative resistor commutes with an action of the group $\mathbf{Z}_2$. This means that

$$I_N(-V) = -I_N(V),$$

which restricts I_N to be an odd function of V. By expanding up to cubic order we get

$$I_N(V) = aV + bV^3,$$

and the physicists say that this makes sense physically only if $b > 0$.

In section 6.2 we derive a set of equations modelling the circuit. The variables in the equations are the voltages measured at two different points in the circuit and the current through the inductor. The parameters are the two capacitances, the coupling resistor and a, b from the expansion of I_N. Then we rescale the quantities obtaining something more convenient for the mathematicians. By more convenient we mean that application of local bifurcation theory involves simpler formulas.

In section 6.3 we analyse the model as much as local methods permit. The origin is always a steady state and by linearizing the system around this state we locate in the parameter space three kinds of bifurcations:

- Pitchfork when one eigenvalue is zero. These occur on a plane.
- Hopf when there is a pair of complex conjugate eigenvalues. These occur on surfaces.
- Takens-Bogdanov when there is a nilpotent linear part. These occur on a line.

In section 6.4 a reduction to a 2-dimensional center manifold (which is contracting in this case) is performed near the Takens-Bogdanov line. By analyzing the reduced system we find a degenerate point whose effect is basically to reverse time in the center manifold. This degeneracy occurs when the two capacitances have the same value.

One of the features of a Takens-Bogdanov point is to imply homoclinic bifurcations nearby (see Guckenheimer and Holmes [35], chapter 7 for a complete description). Such bifurcations are no longer local away from a small neighbourhood of the Takens-Bogdanov point. By analysing the saddle point, Glendinning and Sparrow [30] made some predictions about what happens near homoclinicity in 3-dimensional systems. Glendinning [29] improved these results for systems with $\mathbf{Z}_2$-symmetry. If all eigenvalues are real there is no reason for anything else to happen, but if there is a pair of complex conjugate eigenvalues with a sufficiently strong spin we have a so called Šilnikov bifurcation. This implies a complicated sequence of bifurcations exploring the whole three dimensionality of the phase space, namely infinite sequences of period-doubling and symmetry-breaking bifurcations and more complicated homoclinic orbits. In section 6.6 we divide the parameter space according to Glendinning and Sparrow.

Now if we locate the homoclinic bifurcations we can tell which class they belong to. This is what we do in section 6.6 by numerical simulations. This global analysis is performed in a slice of the parameter space transversal to the Takens-Bogdanov line. The slice is placed at the side of the degenerate point that has more convenient stability.

6.2 Model Equations

Applying Kirchoff's Current Law to the circuit in figure 5.1 and assuming that the characteristic of the negative resistor is an odd function of V_1 truncated at cubic order, we obtain the equations

$$\begin{aligned}
\dot{V}_1 &= -\frac{1}{C_1}\left[bV_1^3 + \left(a + \frac{1}{r}\right)V_1 - \frac{1}{r}V_2\right] \\
\dot{V}_2 &= \frac{1}{C_2}\left[\frac{1}{r}(V_1 - V_2) - I_L\right] \\
\dot{I}_L &= \frac{1}{L}V_2,
\end{aligned} \tag{6.1}$$

where the variables V_1 and V_2 are voltages produced by the circuit, I_L is the current through the inductor, C_1 and C_2 are capacitances and r is the coupling resistor. This system commutes with the action of the group $\mathbf{Z}_2$

$$(V_1, V_2, I_L) \mapsto (-V_1, -V_2, -I_L).$$

For subsequent analysis it is more convenient to rescale as follows:

$$X = \sqrt{br}V_1 \qquad Y = \sqrt{br}V_2 \qquad Z = \sqrt{br^3}I_L \qquad \tau = \frac{1}{rC_2}t$$

obtaining the system in the form

$$\begin{aligned}
\dot{X} &= -\gamma(X^3 - \alpha X - Y) \\
\dot{Y} &= X - Y - Z \\
\dot{Z} &= \beta Y,
\end{aligned} \tag{6.2}$$

where differentiation is with respect to τ and the four parameters are $\alpha = -(1 + ar)$, $\beta = \frac{C_2 r^2}{L}$ and $\gamma = \frac{C_2}{C_1}$. Note that β and γ are positive by definition. In this section we use local methods to find the bifurcation set in the (α, β, γ)-space of the system above. The steady states are given by

$$\begin{aligned} S^0 &= (0,0,0) \\ S^{\pm} &= (\pm\sqrt{\alpha}, 0, \pm\sqrt{\alpha}) \qquad \text{if } \alpha > 0. \end{aligned}$$

Thus, there is a supercritical pitchfork bifurcation giving rise to a symmetric double well potential when α is increased past zero.

6.3 Stability Analysis

The Jacobian matrix of system (6.2) is

$$J(X,Y,Z) = \begin{pmatrix} -\gamma(-\alpha + 3X^2) & \gamma & 0 \\ 1 & -1 & -1 \\ 0 & \beta & 0 \end{pmatrix}.$$

There is a Hopf bifurcation of S^0 corresponding to eigenvalues $\pm i\omega_1$, λ_1 where

$$\begin{aligned} \lambda_1 &= \alpha\gamma - 1 \\ \omega_1^2 &= -\alpha\gamma^2(1 + \alpha), \end{aligned}$$

along the intersection of the parabola

$$\beta = \gamma(1 - \alpha\gamma)(1 + \alpha) \tag{6.3}$$

with the region $-1 < \alpha < 0$. In this region $\lambda_1 < 0$, and thus the system reduces to a 2-dimensional centre manifold near the parabola (6.3). In order to have a 3-dimensional centre manifold we also need $\lambda_1 = 0$. This is prohibited in the present problem since $\lambda_1 = 0$ implies $\alpha = \frac{1}{\gamma} > 0$ which contradicts the condition $-1 < \alpha < 0$.

System (6.2) has a TB bifurcation when the reduced system has a nilpotent linear part. This happens when $\omega_1 = 0$ which occurs for $(\alpha, \beta) = (0, \gamma)$ for each $\gamma > 0$. We will return to this point in section 6.4. A TB bifurcation also occurs for $(\alpha, \beta) = (-1, 0)$. This point lies on the boundary of our domain and must be investigated by approaching the limit $\beta \to 0$. In terms of the physical parameters this limit can be achieved by taking $r \to 0$ or $C_2 \to 0$ or $L \to \infty$. We do not pursue this question further.

We now proceed with the linearization about $S^{\pm}$. These undergo a Hopf bifurcation, with eigenvalues $\pm i\omega_2$, λ_2 where

$$\begin{aligned} \lambda_2 &= -2\alpha\gamma - 1, \\ \omega_2^2 &= 2\alpha\gamma^2(1 - 2\alpha), \end{aligned}$$

along the intersection of the parabola

$$\beta = \gamma(1 + 2\alpha\gamma)(1 - 2\alpha) \tag{6.4}$$

with the region $0 < \alpha < \frac{1}{2}$. Since $\lambda_2 = 0$ implies $\alpha = -\frac{1}{2\gamma} < 0$, we are again unable to obtain a 3-dimensional centre manifold. Note that there are double zero eigenvalues when $\alpha = 0$ or $\frac{1}{2}$; $\alpha = 0$, implying $\beta = \gamma$, coincides with the TB point found above. The case $(\alpha, \beta) = (\frac{1}{2}, 0)$ again requires an investigation of the limit $\beta \to 0$ and is not pursued further.

In order to compute the criticality of the Hopf bifurcation from S^0, we apply the coordinate change

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P \begin{pmatrix} z \\ \overline{z} \\ x \end{pmatrix}$$

such that

$$P^{-1}LP = \begin{pmatrix} i\omega_1 & 0 & 0 \\ 0 & -i\omega_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix},$$

where $L = J(S^0)$. Note that this preserves the $\mathbf{Z}_2$ symmetry

$$(z, x) \mapsto (-z, -x).$$

Every ODE with this symmetry and the correct linear part is

$$\begin{aligned} \dot{z} &= i\omega_1 z + c_0 z^3 + c_1 z^2\overline{z} + c_2 z\overline{z}^2 + c_3\overline{z}^3 + d_0 x z^2 + d_1 x z\overline{z} + d_2 x\overline{z}^2 + \\ &\quad + e_0 x^2 z + e_1 x^2\overline{z} + f_0 x^3 + O(5) \\ \dot{x} &= \lambda_1 x + (h_0 z^3 + \overline{h}_0\overline{z}^3) + (h_1 z^2\overline{z} + \overline{h}_1\overline{z}^2 z) + (j_0 x z^2 + \overline{j}_0 x\overline{z}^2) + \\ &\quad + (k_0 x^2 z + \overline{k}_0 x^2\overline{z}) + l_0 x^3 + O(5). \end{aligned}$$

Changing coordinates once more by

$$\rho = z - \frac{c_0}{2i\omega_1}z^3 + \frac{c_2}{2i\omega_1}z\overline{z}^2 + \frac{c_3}{4i\omega_1}\overline{z}^3,$$

which also preserves the symmetry, and making $x = 0$ we get

$$\dot{\rho} = i\omega_1\rho + c_1\rho^2\overline{\rho} + O(5).$$

So the only important coefficient is $\mathrm{Re}(c_1)$ and the limit cycle created at the Hopf bifurcation is unstable or stable if $\mathrm{Re}(c_1)$ is $>$ or < 0 respectively. This is why c_1 is used to determine criticality. A straightforward calculation gives

$$\mathrm{Re}(c_1) = -\frac{3\omega_1^2\gamma^3(\gamma + 2\alpha\gamma - 1)}{2(\alpha\gamma^2 + 2\alpha\gamma - 1)}.$$

Thus $\mathrm{Re}(c_1) >$ or < 0 if $\alpha >$ or $< \frac{1}{2}\gamma^{-1}(1-\gamma)$, respectively.

In figure 6.1, a partial bifurcation set in the (α, β)-space summarizes these results for $\gamma \geq 1$. It shows the pitchfork P, the Hopf bifurcation H of S^0 and the Hopf bifurcation h from $S^{\pm}$. The point where H changes criticality with respect to the bifurcation parameter β is shown as H^1. Note that H^1 exists if and only if $\gamma \geq 1$ coinciding with TB when $\gamma = 1$. This allows us to predict the existence of a critical Hopf bifurcation h^1 of S^0 somewhere along the parabola h coinciding with TB and H^1 when $\gamma = 1$.

When $\gamma \leq 1$, the phase portraits around the TB point are obtained by changing the stability of all the equilibria in figure 6.1 (see figure 5.4). In this case, the understanding of the behaviour in a small neighbourhood of TB is all we need. Other stable equilibria are not expected no matter how far we go from this point in the parameter space. This is why from now on we concentrate on the case $\gamma > 1$.

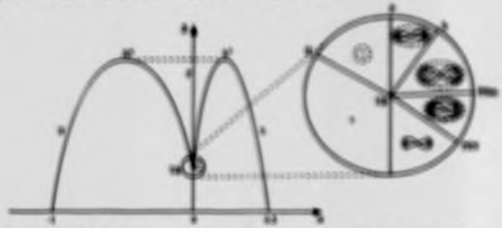


Figure 6.1: Partial bifurcation set obtained by local analysis for $\gamma > 1$. H^1 was computed analytically and it occurs at $(\alpha, \beta) = \left(\frac{1-\gamma}{2\gamma}, [illegible]\right)$. h^1 has not been computed but it was found numerically around $(\alpha, \beta) = \left(\frac{1-\gamma}{3\gamma}, [illegible]\right)$. As γ decreases towards 1, both H^1 and h^1 move towards TB.

6.4 A Two Dimensional Centre Manifold

In this section we make use of results in Guckenheimer and Holmes [35] which we abbreviate here by GH.

We now turn our attention to a study of the neighbourhood of the TB point $(\alpha, \beta) = (0, \gamma)$ where the linearization of system (6.2) about S^0 has a double zero and a negative eigenvalue. The corresponding eigenspaces are

$$\begin{aligned} E_0 &= \{(X, Y, Z) | X - Y - Z = 0\} \\ E_{-1} &= \{(X, Y, Z) | X + \gamma Y = 0, X - Z = 0\}. \end{aligned}$$

Thus the system reduces to a 2-dimensional centre manifold, which is tangent to $X - Y - Z = 0$.

By GH a system with $\mathbf{Z}_2$ symmetry and a double zero eigenvalue reduces to a normal form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= ax^3 + bx^2y \end{aligned} \quad + \quad \mathcal{O}(5) \tag{6.5}$$

in the centre manifold, which is the context of Z_2 symmetry, is satisfied by

$$[illegible] \tag{6.6}$$

if $\gamma \geq 0$. Bifurcation sets of this normal form in the (μ_1, μ_2)-space are given in [illegible] and [illegible]. Comparing with the partial bifurcation set in figure 6.1, we see that in the centre manifold reduction we must obtain a system of the form (6.6) where $a > 0$ and b has the sign of $\gamma - 1$. Recalling that TB bifurcations occur when $\beta = 1$, we expect $(\alpha, \beta) = (0, 1)$ to be a degenerate TB point. [illegible] included in figure 6.1 for the case $\gamma > 1$. [illegible] There remains the question of how these results appear in a global bifurcation picture. We return to this in section 6.6 where we concentrate on the case $\gamma < 1$ since for this case numerical simulations reveal more interesting behaviour.

By computing the third order approximation of the reduced system we get

$$[illegible] \tag{6.7}$$

which is 3-determined if and only if $\gamma \neq 1$. This confirms our prediction of a degeneracy at $\gamma = 1$. Furthermore, we can also confirm that under the imposed condition $\gamma \geq 0$ the only degenerate point is $(\alpha, \beta) = (0, 1)$. In order to obtain its codimension we compute the fifth order approximation of the reduced system for $\gamma = 1$ obtaining

$$[illegible] \tag{6.8}$$

which is fully determined. According to the normal form analysis in GH, this 5-determined system corresponds to a codimension 3 degeneracy.

6.5 Restrictions on Homoclinic Orbits

In this section we make use of results in Glendinning and Sparrow [30] which we abbreviate here by GS.

As shown in section 6.4 by centre manifold techniques, if $\gamma \neq 1$ then system [illegible] exhibits Hopf bifurcations of the three steady states, a homoclinic connection to [illegible] and a saddle-node of limit cycles near a TB point $(\alpha, \beta) = (0, 1)$. As codimension 1 bifurcations, each of these occurs along a line in the (α, β)-space. Information about the Hopf bifurcation lines was given in section 6.3, namely their location in the (α, β)-space and the stability of the created limit cycles was given. The homoclinic line will be followed in section 6.6 by numerical integration. By GS the linearization of the system about the saddle point gives information about behaviour near homoclinicity. In a Z_2 symmetry context, Glendinning [29] shows that if the Šilnikov conditions on the eigenvalues are satisfied, then infinite sequences of saddle-node, period-doubling and symmetry-breaking bifurcations and more complicated homoclinic orbits must exist.

[illegible]

This cusped curve is schematically represented as DE in figure 6.5. It divides the (α, β)-space into two regions according to the number of complex [redacted]. Inside the cusp all three eigenvalues are real and outside there is a complex [redacted] pair.

We also have that any given $\delta \neq \frac{1}{2}$ is constant along the curve

[illegible]

and $\delta = \frac{1}{2}$ if and only if $\alpha = \frac{1}{2}$. The curves $\delta = \frac{1}{2}$ and $\delta = 1$ are also shown schematically in figure 6.5.

We are left with the question of where the homoclinic line goes when it leaves a small neighbourhood of the TB point $(\alpha, \beta) = (0, \frac{1}{2})$. This is one of the [redacted] in section 6.6. That information together with the analysis of the eigenvalues in this section will be enough to show the existence of Šilnikov behaviour near the segment of the principal homoclinic line that is in the region $\delta < 1$.

[illegible]

8.6 Global Results

So far we have used local methods of bifurcation analysis applied to system (8.4). The information obtained there will be completed here by numerical methods. More precisely, the T-B bifurcation found at $(\alpha, \beta) = (0, \gamma)$ for any γ implies a line of homoclinic orbits in the (α, β)-space. We now want to use global methods to determine the evolution and effects of this homoclinic bifurcation far away from the T-B point. Also, given the change of criticality of the Hopf bifurcations H and h, we expect, in the (α, β)-space, lines where the stable and unstable limit cycles undergo saddle-node bifurcations. These lines we call H' and h' respectively, and as global bifurcations they could not be followed by local methods in the previous sections.

In this section, results of numerical integration of model (8.4) are described. The computations were carried out using a fourth order Runge-Kutta algorithm, and most were done on a [illegible] board attached to a personal computer in single, i.e. 32-bit, precision. By this procedure we describe dynamics related to global bifurcations, namely saddle-node of limit cycles, period doubling and collision of attractors. Double precision calculations were used as a check on selected bifurcations.

All the results will be presented in the (α, β)-space for fixed values of γ. Also here we assume $\gamma > 1$ for the reasons stated before. The value $\gamma = 100$ will be used in this section.

The homoclinic bifurcation PH whose existence was proved near $(\alpha, \beta) = (0, \gamma)$ by center manifold techniques and results in Carr and Guckenheimer and Holmes has been followed. Results for the location of this principal homoclinic line in the (α, β)-space are shown in figure 8.3. Also shown in the figure for comparison are the results of the analysis of the linearization at P^* given in section 8.5. We see that the homoclinic line crosses the cusp DE (double eigenvalue) where a pair of eigenvalues changes from and to complex dynamics.

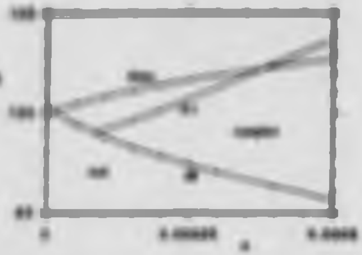


Figure 6.4: [illegible] P^0. [illegible] $\delta = \frac{1}{2}$.

Figure 6.3 also shows the principal homoclinic line going into the region where the Shilnikov condition $\delta < 1$ is satisfied. By Glendinning [30] this implies infinite sequences of saddle-node of limit cycles, period-doubling and symmetry-breaking bifurcations and more complicated homoclinic orbits occurring nearby. These bifurcations occur along codimension 1 lines in the (α, β)-space and therefore they are expected to cross. We have numerical evidence of a codimension 2 degeneracy when a saddle-node of limit cycles and a period-doubling bifurcation undergo a $PD-SN$ interaction. This point is not fully understood. We describe our observations here and keep the problem under investigation. Preliminary results suggest a change of stability of the period-doubling cascade. This also suggests a mechanism for the change of stability near the principal homoclinic bifurcation at $\delta = \frac{1}{2}$.

We proceed by describing some bifurcation sequences obtained by fixing α and decreasing β from a value sufficiently high to catch the whole variety of bifurcations. The line PD in figure 6.4 represents the beginning of a period-doubling cascade bifurcating from the stable limit cycles created at the Hopf bifurcation h or at the saddle-node of limit cycles h^{**} depending on the value assigned to α. A homoclinic bifurcation CH_∞ has been found far from principal homoclinicity. This is a homoclinic bifurcation of one of the limit cycles created in the period-doubling cascade starting at PD. There is a $PD-SN$ interaction where PD meets h^{**}. Also CH_∞ appears to meet this point. This homoclinic bifurcation CH_∞ plays an important role outside the parabola of Hopf bifurcations from $P^{\pm}$. These steady states are stable in that region and P^0 has a 1-dimensional unstable manifold that goes into their basins of attraction. This implies that when the first homoclinic line is crossed by decreasing β, the orbit goes into one of these sinks, rather than into the large stable periodic orbit that has just been created as a result of the homoclinic bifurcation.

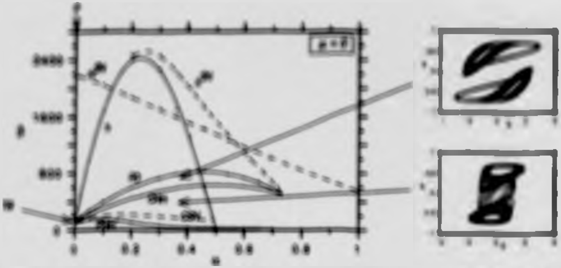


Figure 6.4: Bifurcation set at $\gamma = 100$ and $\rho = 0$. The attractors before and after a symmetry-increasing bifurcation are also shown. Before the symmetry-increasing bifurcation $(\alpha, \beta) = (0.[illegible], 700)$ and after $(\alpha, \beta) = (0.[illegible], [illegible])$.

The stable periodic orbit obtained by the sequence of bifurcations described above dies in a saddle-node of limit cycles along the line CSN. We remark that PD, CHm and CSN do not meet the TB point $(0, \gamma)$. If they did, then this would contradict the existence of a 2-dimensional centre manifold. Healey *et al* [27] have experimental evidence that PD folds over before getting into a sufficiently small neighbourhood of TB and that it stops being observable at a $PD - SN$ interaction.

We now have a picture of how the Hopf and homoclinic bifurcations whose occurrence was shown near $(\alpha, \beta) = (0, \gamma)$ evolve as we leave a small neighbourhood of that point. The saddle-node of limit cycles proved to exist near $(0, \gamma)$ by a centre manifold reduction in section 6.4 has not been followed and is not included in figure 6.4. The bifurcation set now includes some behaviour not associated with $(0, \gamma)$, namely PD, CHm and CSN.

Also in figure 6.4 is the line H^{SN} where the limit cycles created at the Hopf bifurcations H from $S^{\pm}$ undergo a saddle-node bifurcation. The line H^{SN} meets H at a codimension 2 Hopf bifurcation H^*. This point is not shown in the figure but it can be recalled from figure 6.1 which shows the analytical prediction for its location.

Homoclinicities to S^0 have also been found. Figure 6.5 shows a line nHm where these principal homoclinic connections occur in the (α, β)-space. The principal homoclinic line to S^0 is also shown for comparison. We observe that, by the center manifold analysis in section 6.4, nHm does not meet the point $(0, \gamma)$.

The labels corresponding to bifurcations presented up to this point are listed in Table 6.1.

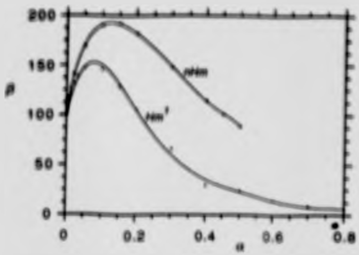


Figure 6.8: Principal homoclinic lines to S^0 and S^a. The homoclinic line PHm to S^0 meets the TB point. The homoclinic line nHm to S^a does not meet this point.

Table 6.1: Notation for bifurcations

P	pitchfork bifurcation
sn	saddle-node bifurcation
H	Hopf bifurcation from S^0
h	Hopf bifurcation from S^a
H'	degenerate H where a change of criticality occurs
TB	Takens-Bogdanov bifurcation
H^{SN}	saddle-node of limit cycles created at H
h^{SN}	saddle-node of limit cycles created at h
PD	period-doubling of a stable limit cycle created at h or h^{SN}
PHm	principal homoclinic bifurcation to S^0
CHm	first homoclinic bifurcation to S^0 crossed by decreasing β (crisis)
nHm	homoclinic bifurcation to S^a
PSN	saddle-node of limit cycles involving one created at PHm
CSN	saddle-node of limit cycles involving one created at CHm

Chapter 7

Breaking the Symmetry

7.1 Introduction

In this chapter we consider the effect of unfolding the characteristic of the negative resistor with a parameter ν as

$$I_N(V) = \nu + aV + bV^3,$$

where $a < 0$ and $b > 0$. Note that the parameter ν breaks the $\mathbf{Z}_2$-symmetry observed in chapter 6. The aim of this chapter is to describe how the results obtained for the symmetric system change by increasing the perturbation ν. As in the previous chapter we use local methods of bifurcation theory when appropriate. However, the absence of the $\mathbf{Z}_2$-symmetry makes the analysis much more complicated and our results are less complete in this case. This is why we are much more dependent on numerical simulations and guesswork.

In section 7.2 a direct calculation gives the steady states to the system. When the ratio of the capacitances is fixed there is a cusped sheet dividing the parameter space according to the number of steady states (one or three). This sheet contains a Takens-Bogdanov cusp found by stability analysis. At the cusp point is the $\mathbf{Z}_2$-symmetric system that was reduced to a 2-dimensional center manifold in chapter 6. Here we describe the effect of the symmetry-breaking parameter on the reduced system. When this parameter is arbitrarily small the bifurcation set is analogue to that obtained by Boissonade and De Kepper [21] for chemical reactions in a continuously stirred tank reactor. We suggest how this picture should evolve when the symmetry-breaking gets large. This prediction is based on numerical simulations (not shown here) and mainly guesswork.

In section 7.3 we keep fixed the ratio of the capacitances at some convenient value and perform numerical simulations on a slice of the parameter space. This slice is taken by fixing the value of the symmetry breaking parameter. The resulting picture is compared with its analogue for the symmetric system.

7.2 Local Analysis and Guesswork

In this section, local methods of bifurcation theory are used to describe some of the behaviour of system

$$\begin{aligned} \dot{X} &= -\gamma(X^3 - \alpha X - Y + \mu) \\ \dot{Y} &= X - Y - Z \\ \dot{Z} &= \beta Y. \end{aligned} \tag{7.1}$$

This set of equations is obtained by applying Kirchoff's Current Law to the circuit in figure 5.1 and scaling the quantities as in chapter 6. The $\mathbf{Z}_2$-symmetry that was present in system (6.2) is now broken with the parameter $\mu = \sqrt{br^3}\nu$, where ν is a new term added to the characteristic of the negative resistor.

As in chapter 6 we fix γ at some value > 1. A bifurcation picture will be given on sections of the (α, β, μ)-space. For each fixed β there is a steady state bifurcation when we cross the cusped sheet $\left(\frac{\mu}{2}\right)^2 = \left(\frac{\alpha}{3}\right)^3$ as in figure 7.1 by increasing α. This bifurcation is a pitchfork if $\mu = 0$ and a saddle-node if $\mu \neq 0$. The state of the system at a saddle-node bifurcation is

$$(X, Y, Z) = \left(\mathrm{sgn}(\mu)\sqrt{\frac{\alpha}{3}}, 0, \mathrm{sgn}(\mu)\sqrt{\frac{\alpha}{3}}\right), \tag{7.2}$$

where $\alpha \geq 0$. Linearizing about these steady states we see that the Jacobian does not depend on α. So for $\mu \neq 0$ there is still a TB bifurcation at $\beta = \gamma$ but now for any (α, μ) along the cusp $(\frac{\mu}{2})^2 = (\frac{\alpha}{3})^3$ as in figure 7.1. Thus, for each fixed $\gamma > 0$ there is a TB cusp in the plane $\beta = \gamma$.

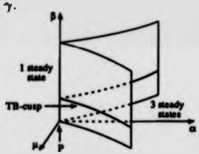


Figure 7.1: Cusped sheet where steady-state bifurcations occur and TB cusp. The steady-state bifurcations are pitchforks along the line P and saddle-node otherwise.

We recall that system (7.1) reduces to a 2-dimensional centre manifold near $(\alpha, \beta, \mu) = (0, \gamma, 0)$, and that system (6.6) in section 6.4 is the normal form for the universal unfolding of the most generic $\mathbf{Z}_2$-symmetric TB singularity. In order to break completely the symmetry of the reduced system (6.6), we need the parameters (μ_1, μ_2). The non-symmetric universal unfolding is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 x^2 + \nu_1 x + \nu_2 y + a x^3 + b x^2 y \end{aligned} \quad + \quad \mathcal{O}(4) \tag{7.3}$$

if $a, b \neq 0$. This family of planar systems is described in Dangelmayr and Guckenheimer [26]. These authors divide the (μ_1, μ_2)-plane into regions bounded by curves along which codimension 3 bifurcations occur. To each of these regions is associated a bifurcation set in the (ν_1, ν_2)-plane. By varying the symmetry-breaking parameter μ of our system, we will be describing a line in the (μ_1, μ_2)-plane. By centre manifold calculations near $(\alpha, \beta, \mu) = (0, \gamma, 0)$ we see that system (7.1) reduces the normal form (7.3) with $\mu_2 = 0$. By a change of coordinates we can show equivalence with a time reverse of the planar Van der Pol-Duffing oscillator presented in section 5.2 and fully described by Guckenheimer [34] in the context of chemical reactors. The 'cross-shaped diagram' given by this author, together with the location of steady-state bifurcations obtained above, gives the bifurcation set in figure 7.2(a) obtained by fixing α arbitrarily close to zero and varying β and μ. Numerical simulations, not shown here in detail, suggest an evolution of this bifurcation set by increasing α as shown schematically in figure 7.2(b,c). By schematic representation we mean not only that the scale is not real but also that the complicated Šilnikov sequence of bifurcations is represented by the single homoclinic line Hm. A planar representation of the phase portraits in each region of figure 7.2 is given in figure 7.3. For simplicity we make the convention that $\mathbf{Z}_2$-conjugate phase portraits are equivalent. See figure 7.5 for a schematic localization of the values of α chosen in figure 7.2 relative to the cusp $(\frac{\mu}{2})^2 = (\frac{\alpha}{3})^3$.

7.3 Numerical Simulations

The aim of this section is to find an analogue to figure 6.4 when the symmetry-breaking parameter μ is large. Figure 7.4 shows the bifurcation set in the (α, β)-space for $(\gamma, \mu) = (100, 0.01)$. Apart from the saddle-node of steady states sn that has been computed analytically all the other lines were obtained by numerical simulations. A comparison with figure 6.4 shows the expected fact (expressed in the diagram of figure 5.2b of chapter 5) that the coincidence of bifurcations from states that are conjugate by the $\mathbf{Z}_2$ symmetry in the idealization $\mu = 0$ no longer coincide when $\mu \neq 0$. On the other hand bifurcations involving only symmetric states when $\mu = 0$ (mapped onto themselves by the symmetry) do not split into two when $\mu \neq 0$.

One of the most remarkable phenomena observed by breaking the $\mathbf{Z}_2$ symmetry is the way that CHm splits into two lines when μ is increased from zero. We recall from chapter 6 that when this line is crossed from above we observe at CHm the collision of two conjugate attractors $C^{\pm}$ with the saddle-focus S^0. The result is a symmetric attractor C^0. The projection of the attractors $C^{\pm}$ and C^0 onto the (X, Z) phase-space was obtained by numerical integration and the result is shown in figure 6.4. When $\mu = 0.01$ and starting from above the two lines CHm we still have two attractors but they are no longer conjugate by any symmetry. Thus there is no longer a reason that they should collide simultaneously with the saddle-focus. Thus by decreasing β when we cross the first CHm only one of the attractors stops being an invariant set. By crossing the second CHm the remaining attractor collides with the saddle-focus and we obtain a larger attractor that may be seen as a perturbation of C^0. A projection of the referred attractors onto the (X, Z)-space is shown in figure 7.4. Again they have been obtained by numerical integration.

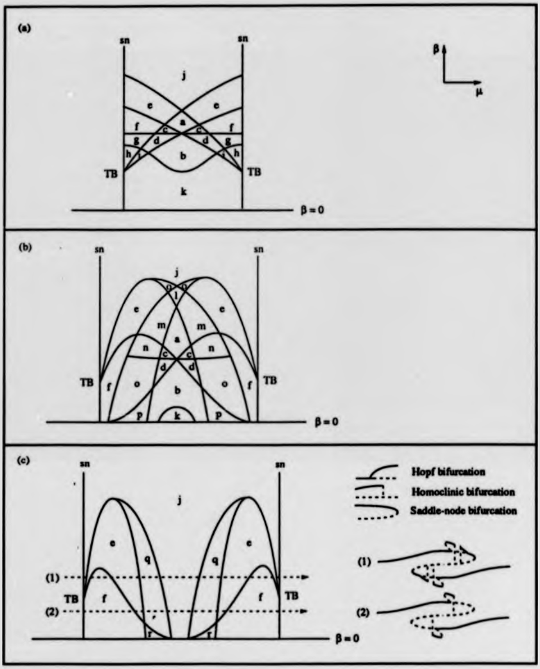


Figure 7.2: Schematic sequence of bifurcation sets for fixed values of α: (a) $\alpha_1 : \alpha_1 \approx 0$; (b) $\alpha_2 : \alpha_2 > \alpha_1$; (c) $\alpha_3 : \alpha_3 > \alpha_2$. See figure 7.5 for a localization of α_j in the (μ, α) plane. Lower case letters correspond to phase portraits from figure 7.3. The bistable region of Guckenheimer cross-shaped diagram can be obtained from (a) by joining the top of the two *sn*-lines. In (c) we show also the bifurcation diagrams by following paths (1) and (2). Note that these paths cross the whole region of bistability by keeping β fixed and increasing μ.

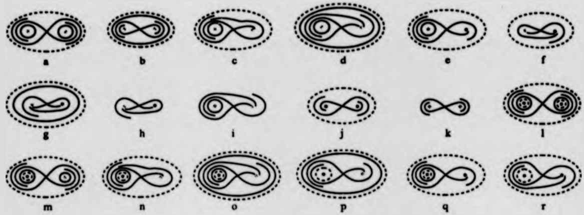


Figure 7.3: Planar phase portraits corresponding to figures 5.5 and 7.2. Crosses are saddle points, black (grey) dots are stable (unstable) steady states and solid (dashed) loops are stable (unstable) limit cycles.

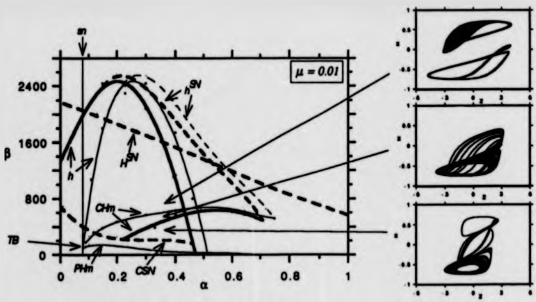


Figure 7.4: Bifurcation set at $\gamma = 100$ and $\mu = 0.01$ (see figure 7.5 for a localization of $\mu = 0$ and $\mu = 0.01$). Also shown are attractors obtained from numerical simulations for parameter values $(\alpha, \beta) = (0.35, 700), (0.35, 510)$ and $(0.35, 300)$.

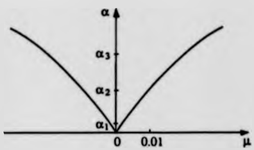


Figure 7.5: Schematic localization of the sectionc taken in figures 6.4, 7.2 and 7.4 relative to the projection of the cusped sheet onto the (μ, α) plane.

Appendix A

Normal Form with $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-Symmetry

In this appendix we consider the normal form for a set of equations

$$f(x, y, z, \lambda) = 0,$$

where $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is equivariant under an action of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and λ is a bifurcation parameter. Such mapping has a two parameter universal unfolding H. The form of H is given and analysed here.

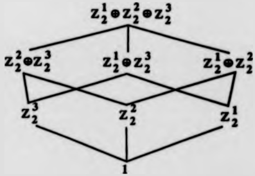
Define an action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ as being generated by

$$\begin{aligned} \tau_1 &: \ (x, y, z) \mapsto (-x, y, z) \\ \tau_2 &: \ (x, y, z) \mapsto (x, -y, z) \\ \tau_3 &: \ (x, y, z) \mapsto (x, y, -z). \end{aligned}$$

The table of isotropy subgroups and fixed point subspaces is

Label	Isotropy subgroup	Fixed point subspace	Dimension
(0)	$\mathbf{Z}_2^1 \oplus \mathbf{Z}_2^2 \oplus \mathbf{Z}_2^3$	$(0,0,0)$	0
(1)	$\mathbf{Z}_2^2 \oplus \mathbf{Z}_2^3$	$(x,0,0)$	1
(2)	$\mathbf{Z}_2^1 \oplus \mathbf{Z}_2^3$	$(0,y,0)$	1
(3)	$\mathbf{Z}_2^1 \oplus \mathbf{Z}_2^2$	$(0,0,z)$	1
(4)	$\mathbf{Z}_2^3$	$(x,y,0)$	2
(5)	$\mathbf{Z}_2^2$	$(x,0,z)$	2
(6)	$\mathbf{Z}_2^1$	$(0,y,z)$	2
(7)	$\mathbf{1}$	(x,y,z)	3

where $\mathbf{Z}_2^j = \langle \tau_j \rangle$. The lattice of isotropy subgroups is



Given a group Γ, we say the a mapping $f(x)$ is Γ-equivariant if and only if it satisfies the condition

$$f(\gamma x) \quad = \quad \gamma f(x),$$

for all elements $\gamma \in \Gamma$. By using the generators of $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ given above, it is a straightforward calculation to see that the general form for an equivariant depending on a bifurcation parameter is

$$\begin{aligned} f_1(x,y,z,\lambda) &= ax \\ f_2(x,y,z,\lambda) &= by \\ f_3(x,y,z,\lambda) &= cz, \end{aligned}$$

where a, b, c are functions of x^2, y^2, z^2 and λ. We proceed by reducing f to normal form.

Proposition 11 *The germ f is $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalent to the normal form*

$$\begin{aligned} h_1(x,y,z,\lambda) &= (\epsilon_1 x^2 + n_1 y^2 + n_2 z^2 + \epsilon_2 \lambda)x \\ h_2(x,y,z,\lambda) &= (n_3 x^2 + \epsilon_3 y^2 + n_4 z^2 + \epsilon_4 \lambda)y \\ h_3(x,y,z,\lambda) &= (n_5 x^2 + n_6 y^2 + \epsilon_5 z^2 + \epsilon_6 \lambda)z, \end{aligned}$$

where

$$\epsilon_1 = sgn(a_{N_1}) \qquad \epsilon_3 = sgn(b_{N_2}) \qquad \epsilon_5 = sgn(c_{N_3})$$

$$\epsilon_2 = sgn(a_\lambda) \qquad \epsilon_4 = sgn(b_\lambda) \qquad \epsilon_6 = sgn(c_\lambda)$$

$$n_1 = \left|\frac{b_\lambda}{b_{N_2} a_\lambda}\right| a_{N_2} \qquad n_3 = \left|\frac{a_\lambda}{a_{N_1} b_\lambda}\right| b_{N_1} \qquad n_5 = \left|\frac{a_\lambda}{a_{N_1} c_\lambda}\right| c_{N_1}$$

$$n_2 = \left|\frac{c_\lambda}{c_{N_3} a_\lambda}\right| a_{N_3} \qquad n_4 = \left|\frac{c_\lambda}{c_{N_3} b_\lambda}\right| b_{N_3} \qquad n_6 = \left|\frac{b_\lambda}{b_{N_2} c_\lambda}\right| c_{N_2},$$

and $N_1 = x^2$, $N_2 = y^2$, $N_3 = z^2$ and λ is the bifurcation parameter.

Proof The 3rd order truncation of the germ f is

$$\begin{aligned} f_1(x,y,z,\lambda) &= (a_{N_1}x^2 + a_{N_2}y^2 + a_{N_3}z^2 + a_\lambda\lambda)x \\ f_2(x,y,z,\lambda) &= (b_{N_1}x^2 + b_{N_2}y^2 + b_{N_3}z^2 + b_\lambda\lambda)y \\ f_3(x,y,z,\lambda) &= (c_{N_1}x^2 + c_{N_2}y^2 + c_{N_3}z^2 + c_\lambda\lambda)z. \end{aligned}$$

By Golubitsky *et al.* [11], a germ h is $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalent to f up to 3rd order if and only if

$$h(x,y,z,\lambda) = Sf(X,\Lambda) \tag{A.1}$$

where

$$\begin{aligned} &X(x,y,z,\lambda) = (Ax, By, Cz) \\ &\Lambda(\lambda) = \sigma\lambda \\ &S(x,y,z,\lambda) = \begin{pmatrix} D & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix} \end{aligned}$$

and A, B, C, D, E, F, σ are positive constants. By substituting X, Λ and S in (A.1) we get

$$\begin{aligned} h_1(x,y,z,\lambda) &= (a_{N_1}A^3Dx^2 + a_{N_2}AB^2Dy^2 + a_{N_3}AC^2Dz^2 + a_\lambda\sigma AD\lambda)x \\ h_2(x,y,z,\lambda) &= (b_{N_1}A^2BEx^2 + b_{N_2}B^3Ey^2 + b_{N_3}BC^2Ez^2 + b_\lambda\sigma BE\lambda)y \\ h_3(x,y,z,\lambda) &= (c_{N_1}A^2CFx^2 + c_{N_2}B^2CFy^2 + c_{N_3}C^3Fz^2 + c_\lambda\sigma CF\lambda)z. \end{aligned}$$

By imposing the conditions

$$\begin{aligned} &|a_{N_1}|A^3D = 1 \qquad |b_{N_2}|B^3E = 1 \qquad |c_{N_3}|C^3F = 1 \\ &|a_\lambda|\sigma AD = 1 \qquad |b_\lambda|\sigma BE = 1 \qquad |c_\lambda|\sigma CF = 1 \end{aligned}$$

we get the required result. □

The normal form h has a $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-universal unfolding as

$$\begin{aligned} H_1(x,y,z,\lambda,\tilde{n}_1,\ldots,\tilde{n}_6,\tilde{r}_1,\tilde{r}_2) &= (\epsilon_1x^2 + \tilde{n}_1y^2 + \tilde{n}_2z^2 + \epsilon_2\lambda)x \\ H_2(x,y,z,\lambda,\tilde{n}_1,\ldots,\tilde{n}_6,\tilde{r}_1,\tilde{r}_2) &= (\tilde{n}_3x^2 + \epsilon_3y^2 + \tilde{n}_4z^2 + \epsilon_4\lambda + \tilde{r}_1)y \\ H_3(x,y,z,\lambda,\tilde{n}_1,\ldots,\tilde{n}_6,\tilde{r}_1,\tilde{r}_2) &= (\tilde{n}_5x^2 + \tilde{n}_6y^2 + \epsilon_5z^2 + \epsilon_6\lambda + \tilde{r}_2)z, \end{aligned}$$

where $(\tilde{n}_1,\ldots,\tilde{n}_6,\tilde{r}_1,\tilde{r}_2)$ varies on a neighbourhood of $(n_1,\ldots,n_6,0,0)$.

We proceed by analysing bifurcations of the set of equations

$$H(x,y,z,\lambda,\tilde{n},\tilde{r}) = 0$$

for the bifurcation parameter λ. The coefficients $\tilde{n}_j$ are the moduli parameters and $\tilde{r}_j$ are the unfolding parameters.

The branching equations and eigenvalues are as

Label	Branching equations	Signs of eigenvalues
(1)	$\lambda = -\epsilon_1\epsilon_2 x^2$	ϵ_1 $n_3x^2 + \epsilon_4\lambda + \tilde{r}_1$ $n_5x^2 + \epsilon_6\lambda + \tilde{r}_2$
(2)	$\lambda = -\epsilon_4\tilde{r}_1 - \epsilon_3\epsilon_4 y^2$	$n_1y^2 + \epsilon_2\lambda$ ϵ_3 $n_6y^2 + \epsilon_6\lambda + \tilde{r}_2$
(3)	$\lambda = -\epsilon_6\tilde{r}_2 - \epsilon_5\epsilon_6 z^2$	$n_2z^2 + \epsilon_2\lambda$ $n_4z^2 + \epsilon_4\lambda + \tilde{r}_1$ ϵ_5
(4)	$\lambda = -\epsilon_1\epsilon_2x^2 - n_1\epsilon_2y^2$ $= -\epsilon_4\tilde{r}_1 - n_3\epsilon_4x^2 - \epsilon_3\epsilon_4y^2$	$\mathrm{tr} = \epsilon_1x^2 + \epsilon_3y^2$ $\det = (\epsilon_1\epsilon_3 - n_1n_3)x^2y^2$ $n_5x^2 + n_6y^2 + \epsilon_6\lambda + \tilde{r}_2$
(5)	$\lambda = -\epsilon_1\epsilon_2x^2 - n_2\epsilon_2z^2$ $= -\epsilon_6\tilde{r}_1 - n_5\epsilon_6x^2 - \epsilon_5\epsilon_6z^2$	$\mathrm{tr} = \epsilon_1x^2 + \epsilon_5z^2$ $n_3x^2 + n_4z^2 + \epsilon_4\lambda + \tilde{r}_1$ $\det = (\epsilon_1\epsilon_5 - n_2n_5)x^2z^2$
(6)	$\lambda = -\epsilon_4\tilde{r}_1 - \epsilon_3\epsilon_4y^2 - n_4\epsilon_4z^2$ $= -\epsilon_6\tilde{r}_2 - n_6\epsilon_6y^2 - \epsilon_5\epsilon_6z^2$	$n_1y^2 + n_2z^2 + \epsilon_2\lambda$ $\mathrm{tr} = \epsilon_3y^2 + \epsilon_5z^2$ $\det = (\epsilon_3\epsilon_5 - n_4n_6)y^2z^2$
(7)	$\lambda = -\epsilon_1\epsilon_2x^2 - n_1\epsilon_2y^2 - n_2\epsilon_2z^2$ $= -\epsilon_4\tilde{r}_1 - n_3\epsilon_4x^2 - \epsilon_3\epsilon_4y^2 - n_4\epsilon_4z^2$ $= -\epsilon_6\tilde{r}_2 - n_5\epsilon_6x^2 - n_6\epsilon_6y^2 - \epsilon_5\epsilon_6z^2$	eigenvalues of $\begin{matrix} \epsilon_1x^2 & n_1xy & n_2xz \\ n_3xy & \epsilon_3y^2 & n_4yz \\ n_5xz & n_6yz & \epsilon_5z^2 \end{matrix}$

The system is nondegenerate if and only if the following expressions do not vanish

Label	Nondegeneracy condition
(a)	$\epsilon_2\epsilon_3\epsilon_4 - n_1$
(b)	$\epsilon_2\epsilon_5\epsilon_6 - n_2$
(c)	$\epsilon_1\epsilon_2\epsilon_4 - n_3$
(d)	$\epsilon_4\epsilon_5\epsilon_6 - n_4$
(e)	$\epsilon_1\epsilon_2\epsilon_6 - n_5$
(f)	$\epsilon_3\epsilon_4\epsilon_6 - n_6$
(g)	$\epsilon_1\epsilon_3 - n_1 n_3$
(h)	$\epsilon_1\epsilon_5 - n_2 n_5$
(i)	$\epsilon_3\epsilon_5 - n_4 n_6$
(j)	$n_5(\epsilon_2\epsilon_3 - \epsilon_4 n_1) + n_6(\epsilon_1\epsilon_4 - \epsilon_2 n_3) - \epsilon_6(\epsilon_1\epsilon_3 - n_1 n_3)$
(k)	$n_3(\epsilon_2\epsilon_5 - \epsilon_6 n_2) + n_4(\epsilon_1\epsilon_6 - \epsilon_2 n_5) - \epsilon_4(\epsilon_1\epsilon_5 - n_2 n_5)$
(l)	$n_1(\epsilon_4\epsilon_5 - \epsilon_6 n_4) + n_2(\epsilon_3\epsilon_6 - \epsilon_4 n_6) - \epsilon_2(\epsilon_3\epsilon_5 - n_4 n_6)$
(m)	$\epsilon_1\epsilon_3\epsilon_5 - \epsilon_5 n_1 n_3 - \epsilon_3 n_2 n_5 - \epsilon_1 n_4 n_6 + n_2 n_3 n_6 + n_1 n_4 n_5$

From now on we assume that the trivial branch $(x, y, z) = (0, 0, 0)$ is stable before bifurcations occur

$$\epsilon_2 = \epsilon_4 = \epsilon_6 = -1$$

and the primary bifurcations are subcritical

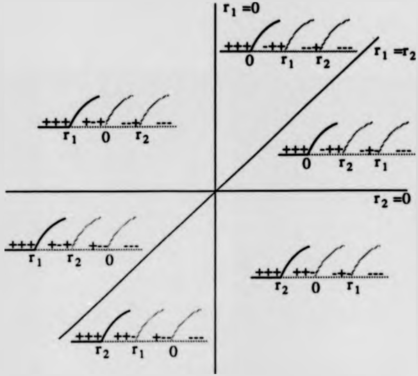
$$\epsilon_1 = \epsilon_3 = \epsilon_5 = 1.$$

This choice is the most likely to occur in physical applications. However, for other values of the ϵ_j the procedure is analogous.

Primary bifurcations occur at the following values of λ

From	To	Critical λ
(0)	(1)	$\lambda_{01} = 0$
(0)	(2)	$\lambda_{02} = \bar{r}_1$
(0)	(3)	$\lambda_{03} = \bar{r}_2$

Thus, the primary branches depend on $\bar{r}_1, \bar{r}_2$ as



Under the assumption that all primary bifurcations are supercritical, we have secondary bifurcations at the following values of λ if the associated conditions are satisfied.

From	To	Critical λ	Bifurcation iff Supercritical iff
(1)	(4)	$\lambda_{14} = \lambda_{01} + \frac{1}{(c)}(\lambda_{02} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{02} - \lambda_{01}) = \mathrm{sgn}(c)$ $\mathrm{sgn}(\lambda_{02} - \lambda_{01}) = \mathrm{sgn}(g)$
(1)	(5)	$\lambda_{15} = \lambda_{01} + \frac{1}{(e)}(\lambda_{03} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{03} - \lambda_{01}) = \mathrm{sgn}(e)$ $\mathrm{sgn}(\lambda_{03} - \lambda_{01}) = \mathrm{sgn}(h)$
(2)	(4)	$\lambda_{24} = \lambda_{02} + \frac{1}{(a)}(\lambda_{01} - \lambda_{02})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{02}) = \mathrm{sgn}(a)$ $\mathrm{sgn}(\lambda_{01} - \lambda_{02}) = \mathrm{sgn}(g)$
(2)	(6)	$\lambda_{26} = \lambda_{02} + \frac{1}{(f)}(\lambda_{03} - \lambda_{02})$	$\mathrm{sgn}(\lambda_{03} - \lambda_{02}) = \mathrm{sgn}(f)$ $\mathrm{sgn}(\lambda_{03} - \lambda_{02}) = \mathrm{sgn}(i)$
(3)	(5)	$\lambda_{35} = \lambda_{03} + \frac{1}{(b)}(\lambda_{01} - \lambda_{03})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{03}) = \mathrm{sgn}(b)$ $\mathrm{sgn}(\lambda_{01} - \lambda_{03}) = \mathrm{sgn}(h)$
(3)	(6)	$\lambda_{36} = \lambda_{03} + \frac{1}{(d)}(\lambda_{02} - \lambda_{03})$	$\mathrm{sgn}(\lambda_{02} - \lambda_{03}) = \mathrm{sgn}(d)$ $\mathrm{sgn}(\lambda_{02} - \lambda_{03}) = \mathrm{sgn}(i)$

Tertiary bifurcations occur at the following values of λ if the associated conditions are satisfied.

From	To	Critical λ	Bifurcation iff
(4)	(7)	$\lambda_{47} = \lambda_{14} + \frac{(e)(g)}{(j)}(\lambda_{15} - \lambda_{14})$	$\mathrm{sgn}(\lambda_{02} - \lambda_{01}) = \mathrm{sgn}(c)$ or $-\mathrm{sgn}(a)$
		$= \lambda_{24} + \frac{(f)(g)}{(j)}(\lambda_{26} - \lambda_{24})$	and $\mathrm{sgn}(\lambda_{02} - \lambda_{01}) = \mathrm{sgn}(c) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{15} - \lambda_{14}) = \mathrm{sgn}(c)\mathrm{sgn}(e)\mathrm{sgn}(j)$
			and $\mathrm{sgn}(\lambda_{01} - \lambda_{02}) = \mathrm{sgn}(a) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{26} - \lambda_{24}) = \mathrm{sgn}(a)\mathrm{sgn}(f)\mathrm{sgn}(j)$
(5)	(7)	$\lambda_{57} = \lambda_{15} + \frac{(c)(h)}{(k)}(\lambda_{14} - \lambda_{15})$	$\mathrm{sgn}(\lambda_{03} - \lambda_{01}) = \mathrm{sgn}(e)$ or $-\mathrm{sgn}(b)$
		$= \lambda_{35} + \frac{(d)(h)}{(k)}(\lambda_{36} - \lambda_{35})$	and $\mathrm{sgn}(\lambda_{03} - \lambda_{01}) = \mathrm{sgn}(e) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{14} - \lambda_{15}) = \mathrm{sgn}(c)\mathrm{sgn}(e)\mathrm{sgn}(k)$
			and $\mathrm{sgn}(\lambda_{01} - \lambda_{03}) = \mathrm{sgn}(b) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{36} - \lambda_{35}) = \mathrm{sgn}(b)\mathrm{sgn}(d)\mathrm{sgn}(k)$
(6)	(7)	$\lambda_{67} = \lambda_{26} + \frac{(a)(i)}{(l)}(\lambda_{24} - \lambda_{26})$	$\mathrm{sgn}(\lambda_{03} - \lambda_{02}) = \mathrm{sgn}(f)$ or $-\mathrm{sgn}(d)$
		$= \lambda_{36} + \frac{(b)(i)}{(l)}(\lambda_{35} - \lambda_{36})$	and $\mathrm{sgn}(\lambda_{03} - \lambda_{02}) = \mathrm{sgn}(f) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{24} - \lambda_{26}) = \mathrm{sgn}(a)\mathrm{sgn}(f)\mathrm{sgn}(l)$
			and $\mathrm{sgn}(\lambda_{02} - \lambda_{03}) = \mathrm{sgn}(d) \Rightarrow$
			$\Rightarrow \mathrm{sgn}(\lambda_{35} - \lambda_{36}) = \mathrm{sgn}(b)\mathrm{sgn}(d)\mathrm{sgn}(l)$

The information in the tables above is enough to draw the bifurcation diagrams given the values of n_j. The possible cases are not listed here because they are too many. However, in chapter 4 on an application to the Bénard convection problem, we show how to draw the bifurcation diagrams by reading these tables.

Appendix B

Normal Form with $Z_2 \oplus D_4$-Symmetry

In this appendix we consider the normal form for a set of equations

$$f(x, y, z, \lambda) = 0,$$

where $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is equivariant under an action of the group $\mathbf{Z}_2 \oplus \mathbf{D}_4$ and λ is a bifurcation parameter. Such mapping has a one parameter universal unfolding H. The form of H is given and analysed here. In section B.1 the $\mathbf{S}_2$-symmetry of H is broken with an additional unfolding parameter and tables containing all the information about existence and criticality of bifurcating branches are given.

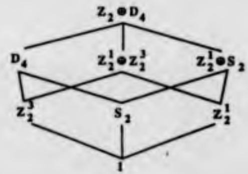
Define an action of $\mathbf{Z}_2 \oplus \mathbf{D}_4$ as being generated by

$$\begin{aligned}
\tau_1 : &\quad (x, y_1, y_2) \mapsto (-x, y_1, y_2) \\
\tau_2 : &\quad (x, y_1, y_2) \mapsto (x, -y_1, y_2) \\
s : &\quad (x, y_1, y_2) \mapsto (x, y_2, y_1).
\end{aligned}$$

The table of isotropy subgroups and fixed point subspaces is follows:

Label	Isotropy subgroup	Fixed point subspace	Dimension
(0)	$\mathbf{Z}_2 \oplus \mathbf{D}_4$	$(0, 0, 0)$	0
(1)	$\mathbf{D}_4$	$(x, 0, 0)$	1
(2)	$\mathbf{Z}_2^1 \oplus \mathbf{Z}_2^3$	$(0, y_1, 0)$	1
(3)	$\mathbf{Z}_2^1 \oplus \mathbf{S}_2$	$(0, y_1, y_1)$	1
(4)	$\mathbf{Z}_2^3$	$(x, y_1, 0)$	2
(5)	$\mathbf{S}_2$	(x, y_1, y_1)	2
(6)	$\mathbf{Z}_2^1$	$(0, y_1, y_2)$	2
(7)	$\mathbf{1}$	(x, y_1, y_2)	3

where $\mathbf{Z}_2^j = \langle \tau_j \rangle$. The lattice of isotropy subgroups is



By a direct calculation using the generators of $\mathbf{Z}_2 \oplus \mathbf{D}_4$ given above, we get the general form of an equivariant depending on a bifurcation parameter as

$$\begin{aligned}
f_1(x, y_1, y_2, \lambda) &= ax \\
f_2(x, y_1, y_2, \lambda) &= (b + c\delta)y_1 \\
f_3(x, y_1, y_2, \lambda) &= (b - c\delta)y_2,
\end{aligned}$$

where a, b, c are functions of x^2, $y_1^2 + y_2^2$, $(y_2^2 - y_1^2)^2$ and λ; and $\delta = y_2^2 - y_1^2$. We proceed by reducing f to normal form.

Proposition 12 *The germ f is $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivalent to the normal form*

$$\begin{aligned}
h_1(x, y_1, y_2, \lambda) &= [\epsilon_1 x^2 + n_1(y_1^2 + y_2^2) + \epsilon_2\lambda]x \\
h_2(x, y_1, y_2, \lambda) &= [n_2 x^2 + n_3(y_1^2 + y_2^2) + \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda]y_1 \\
h_3(x, y_1, y_2, \lambda) &= [n_2 x^2 + n_3(y_1^2 + y_2^2) - \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda]y_2,
\end{aligned}$$

where

$$\epsilon_1 = sgn(a_{N_1}) \qquad \epsilon_2 = sgn(a_\lambda) \qquad \epsilon_3 = sgn(c) \qquad \epsilon_4 = sgn(b_\lambda)$$

$$n_1 = \left|\frac{b_\lambda}{ca_\lambda}\right| a_{N_2} \qquad n_2 = \left|\frac{a_\lambda}{a_{N_1} b_\lambda}\right| b_{N_1} \qquad n_3 = \left|\frac{1}{c}\right| b_{N_2},$$

and $N_1 = x^2$, $N_2 = y_1^2 + y_2^2$ and λ is the bifurcation parameter.

Proof The 3rd order truncation of the germ f is

$$\begin{aligned}
f_1(x, y_1, y_2, \lambda) &= [a_{N_1} x^2 + a_{N_2}(y_1^2 + y_2^2) + a_\lambda \lambda]x \\
f_2(x, y_1, y_2, \lambda) &= [b_{N_1} x^2 + b_{N_2}(y_1^2 + y_2^2) + c(y_2^2 - y_1^2) + b_\lambda \lambda]y_1 \\
f_3(x, y_1, y_2, \lambda) &= [b_{N_1} x^2 + b_{N_2}(y_1^2 + y_2^2) - c(y_2^2 - y_1^2) + b_\lambda \lambda]y_2.
\end{aligned}$$

A germ h is $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivalent to f up to 3rd order if and only if

$$h(x, y_1, y_2, \lambda) = Sf(X, \Lambda) \tag{B.1}$$

where

$$\begin{aligned} X(x, y_1, y_2, \lambda) &= (Ax, By_1, By_2) \\ \Lambda(\lambda) &= \sigma\lambda \\ S(x, y_1, y_2, \lambda) &= \begin{pmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \end{aligned}$$

and A, B, C, D, σ are positive constants. By substituting X, Λ and S in (B.1) we get

$$\begin{aligned} h_1(x, y_1, y_2, \lambda) &= (a_{N_1} A^3 C x^2 + a_{N_2} A B^2 C(y_1^2 + y_2^2) + a_\lambda \sigma A C \lambda) x \\ h_2(x, y_1, y_2, \lambda) &= (b_{N_1} A^2 B D x^2 + b_{N_2} B^3 D(y_1^2 + y_2^2) + c B^3 D(y_2^2 - y_1^2) + b_\lambda \sigma B D \lambda) y_1 \\ h_3(x, y_1, y_2, \lambda) &= (b_{N_1} A^2 B D x^2 + b_{N_2} B^3 D(y_1^2 + y_2^2) - c B^3 D(y_2^2 - y_1^2) + b_\lambda \sigma B D \lambda) y_2. \end{aligned}$$

By imposing the conditions

$$|a_{N_1}| A^3 C = 1 \qquad |a_\lambda| \sigma A C = 1 \qquad |c| B^3 D = 1 \qquad |b_\lambda| \sigma B D = 1$$

we get the required result. □

The normal form h has a $\mathbf{Z}_2 \oplus \mathbf{D}_4$-universal unfolding as follows:

$$\begin{aligned} H_1(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1) &= [\epsilon_1 x^2 + \tilde{n}_1(y_1^2 + y_2^2) + \epsilon_2 \lambda] x \\ H_2(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1) &= [\tilde{n}_2 x^2 + \tilde{n}_3(y_1^2 + y_2^2) + \epsilon_3(y_2^2 - y_1^2) + \epsilon_4 \lambda + \tilde{\rho}_1] y_1 \\ H_3(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1) &= [\tilde{n}_2 x^2 + \tilde{n}_3(y_1^2 + y_2^2) - \epsilon_3(y_2^2 - y_1^2) + \epsilon_4 \lambda + \tilde{\rho}_1] y_2, \end{aligned}$$

where $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1)$ varies on a neighbourhood of $(n_1, n_2, n_3, 0)$.

We proceed by analysing bifurcations of the set of equations

$$H(x, y, z, \lambda, \tilde{n}, \tilde{\rho}) = 0$$

for the bifurcation parameter λ. The coefficients $\tilde{n}_j$ are the moduli parameters and $\tilde{\rho}_1$ is the unfolding parameter.

The branching equations and eigenvalues are as

Label	Branching equations	Signs of eigenvalues
(1)	$\lambda = -\epsilon_1\epsilon_2 x^2$	ϵ_1 $n_2x^2 + \epsilon_4\lambda + \tilde{\rho}_1$ [twice]
(2)	$\lambda = -\epsilon_4\tilde{\rho}_1 - \epsilon_4(n_3 - \epsilon_3)y_1^2$	$n_1y_1^2 + \epsilon_2\lambda$ $n_3 - \epsilon_3$ ϵ_3
(3)	$\lambda = -\epsilon_4\tilde{\rho}_1 - 2\epsilon_4 n_3 y_1^2$	$2n_1y_1^2 + \epsilon_2\lambda$ n_3 $-\epsilon_3$
(4)	$\lambda = -\epsilon_1\epsilon_2x^2 - \epsilon_2 n_1 y_1^2$ $= -\epsilon_4\tilde{\rho}_1 - \epsilon_4 n_2 x^2 - \epsilon_4(n_3 - \epsilon_3)y_1^2$	$\mathrm{tr} = \epsilon_1x^2 + (n_3 - \epsilon_3)y_1^2$ $\det = [\epsilon_1(n_3 - \epsilon_3) - n_1n_2]x^2y_1^2$ ϵ_3
(5)	$\lambda = -\epsilon_1\epsilon_2x^2 - 2\epsilon_2 n_1 y_1^2$ $= -\epsilon_4\tilde{\rho}_1 - \epsilon_4 n_2 x^2 - 2\epsilon_4 n_3 y_1^2$	$\mathrm{tr} = \epsilon_1x^2 + 2n_3y_1^2$ $\det = 2(\epsilon_1n_3 - n_1n_2)x^2y_1^2$ $-\epsilon_3$

The system is nondegenerate if the following expressions do not vanish

Label	Nondegeneracy condition
(a)	$\epsilon_1\epsilon_2n_2 - \epsilon_4$
(b)	n_3
(c)	$n_3 - \epsilon_3$
(d)	$n_3 - \epsilon_2\epsilon_4n_1$
(e)	$\epsilon_4n_1 - \epsilon_2(n_3 - \epsilon_3)$
(f)	$n_3 - \epsilon_1n_1n_2$
(g)	$\epsilon_1(n_3 - \epsilon_3) - n_1n_2$

From now on we assume that the trivial branch $(x, y_1, y_2) = (0,0,0)$ is stable before bifurcations occur

$$\epsilon_2 = \epsilon_4 = -1.$$

We assume also that

$$\epsilon_1 = \epsilon_3 = 1.$$

Primary bifurcations occur at the values of λ in the table below. Branch (1) is always supercritical under our assumptions on the ϵ_j. Conditions for supercriticality of branches (2) and (3) are also given.

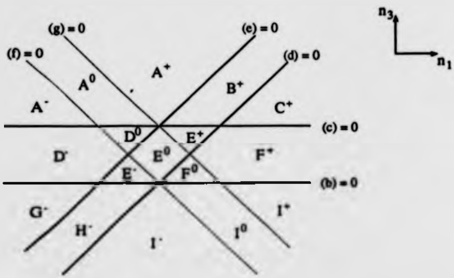
From	To	Critical λ	Supercritical iff
(0)	(1)	$\lambda_{01} = 0$	
(0)	(2)	$\lambda_{02} = \tilde{\rho}_1$	$(c) > 0$
(0)	(3)	$\lambda_{03} = \tilde{\rho}_1$	$(b) > 0$

Secondary bifurcations are at the following values of λ if the associated conditions are satisfied.

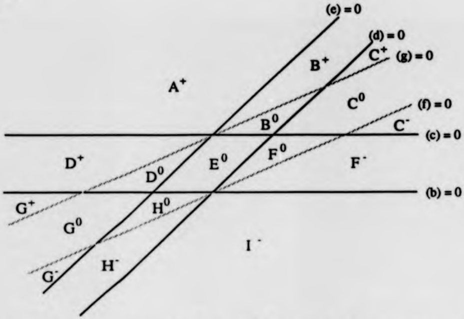
From	To	Critical λ	Bifurcation iff
(1)	(4)	$\lambda_{14} = \lambda_{01} + \frac{1}{(a)}(\lambda_{02} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{02} - \lambda_{01}) = \mathrm{sgn}(a)$
(1)	(5)	$\lambda_{15} = \lambda_{01} + \frac{1}{(a)}(\lambda_{03} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{03} - \lambda_{01}) = \mathrm{sgn}(a)$
(2)	(4)	$\lambda_{24} = \lambda_{02} + \frac{(a)}{(e)}(\lambda_{01} - \lambda_{02})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{02}) = \mathrm{sgn}(e)$
(3)	(5)	$\lambda_{35} = \lambda_{03} - \frac{(b)}{(d)}(\lambda_{01} - \lambda_{03})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{03}) = \mathrm{sgn}(d)$

We proceed by identifying the distinct moduli regions and drawing the bifurcation diagrams corresponding to some of then. For $n_2 < 1$ ($\Leftrightarrow (a) > 0$) there are 27 distinct regions. They are found by considering two distinct cross sections of constant n_2, one with negative constant and the other positive.

$n_2 < 0$

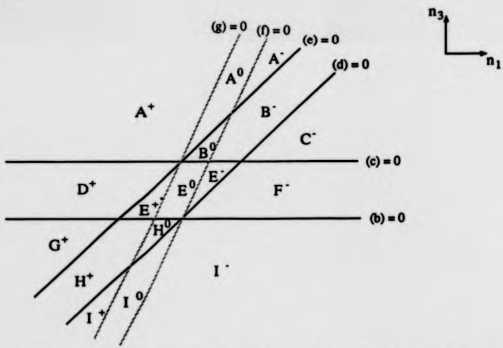


$0 < n_2 < 1$



For $n_2 > 1$ ($\Leftrightarrow$ $(a) < 0$) there are 17 distinct regions. A cross section of constant n_2 is as follows:

$n_2 > 1$

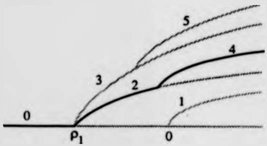


Thus, the total of distinct regions is 44. Now we draw the bifurcation diagrams for $(a) > 0$. If $(a) < 0$, they are obtained in the same way. The tables given above contain all the necessary information, we only need to read them.

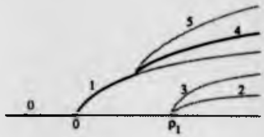
A: (b)>0, (c)>0, (d)>0, (e)>0

A^+ : (f)>0, (g)>0

$\rho_1 < 0$

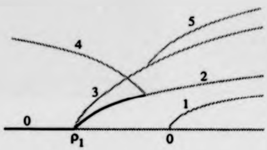


$\rho_1 > 0$

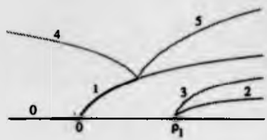


A^0 : (f)>0, (g)<0

$\rho_1 < 0$

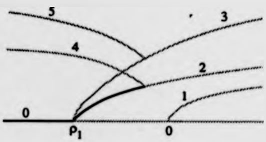


$\rho_1 > 0$

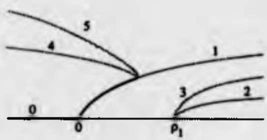


A^- : (f)<0, (g)<0

$\rho_1 < 0$



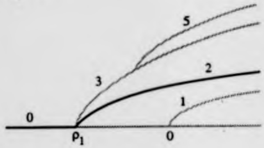
$\rho_1 > 0$



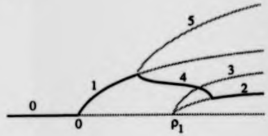
B: (b)>0, (c)>0, (d)>0, (e)<0

B^+ : (f)>0, (g)>0

$\rho_1 < 0$

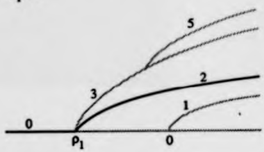


$\rho_1 > 0$

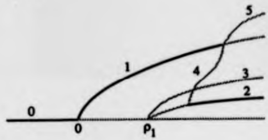


B^0 : (f)>0, (g)<0

$\rho_1 < 0$

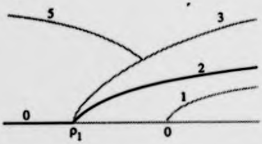


$\rho_1 > 0$

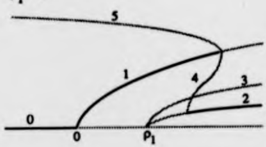


B^- : (f)<0, (g)<0

$\rho_1 < 0$



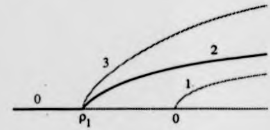
$\rho_1 > 0$



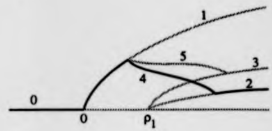
C: (b)>0, (c)>0, (d)<0, (e)<0

C^+ : (f)>0, (g)>0

$\rho_1 < 0$

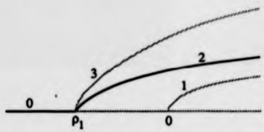


$\rho_1 > 0$

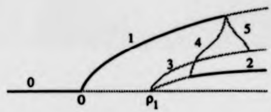


C^0 : (f)>0, (g)<0

$\rho_1 < 0$

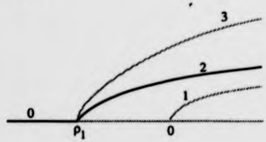


$\rho_1 > 0$

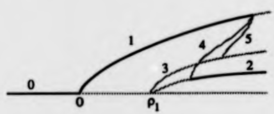


C^- : (f)<0, (g)<0

$\rho_1 < 0$



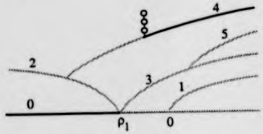
$\rho_1 > 0$



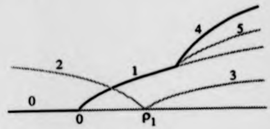
D: (b)>0, (c)<0, (d)>0, (e)>0

D^+ : (f)>0, (g)>0

$\rho_1 < 0$

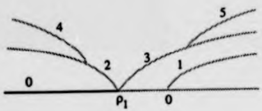


$\rho_1 > 0$

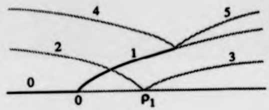


D^0 : (f)>0, (g)<0

$\rho_1 < 0$

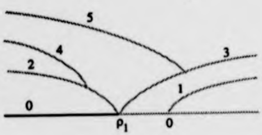


$\rho_1 > 0$

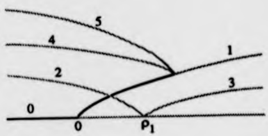


D^- : (f)<0, (g)<0

$\rho_1 < 0$



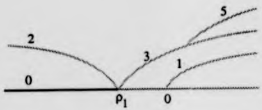
$\rho_1 > 0$



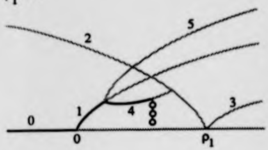
E: (b)>0, (c)<0, (d)>0, (e)<0

E^+ : (f)>0, (g)>0

$\rho_1 < 0$

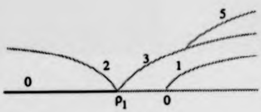


$\rho_1 > 0$

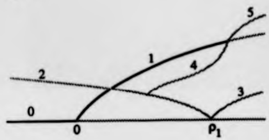


E^0 : (f)>0, (g)<0

$\rho_1 < 0$

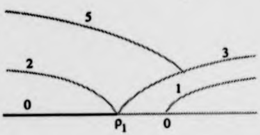


$\rho_1 > 0$

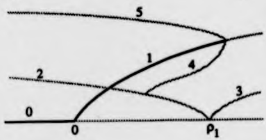


E^- : (f)<0, (g)<0

$\rho_1 < 0$



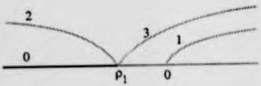
$\rho_1 > 0$



F: (b)>0, (c)<0, (d)<0, (e)<0

F^+ : (f)>0, (g)>0

$\rho_1 < 0$



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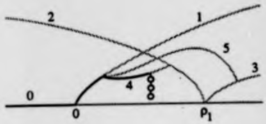
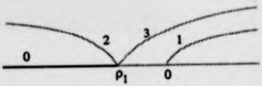
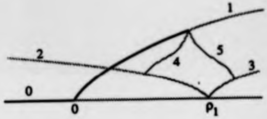


F^0 : (f)>0, (g)<0

$\rho_1 < 0$

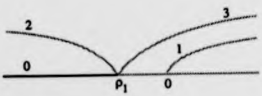


$\rho_1 > 0$

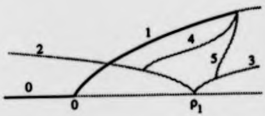


F^- : (f)<0, (g)<0

$\rho_1 < 0$



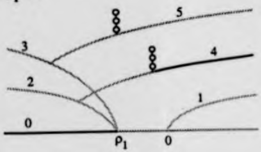
$\rho_1 > 0$



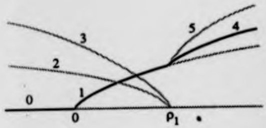
G: (b)<0, (c)<0, (d)>0, (e)>0

G^+ : (f)>0, (g)>0

$\rho_1 < 0$

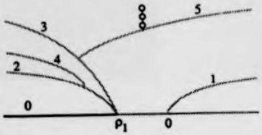


$\rho_1 > 0$



G^0 : (f)>0, (g)<0

$\rho_1 < 0$



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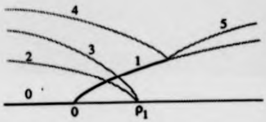
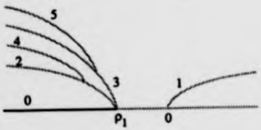


G^- : (f)<0, (g)<0

$\rho_1 < 0$



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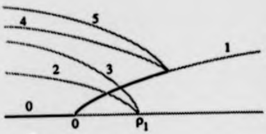
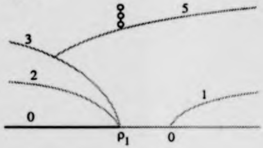


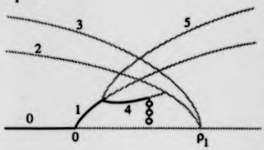
H: (b)<0, (c)<0, (d)>0, (e)<0

H^+ : (f)>0, (g)>0

$\rho_1 < 0$

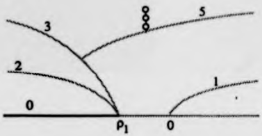


$\rho_1 > 0$

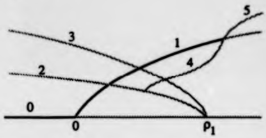


H^0 : (f)>0, (g)<0

$\rho_1 < 0$

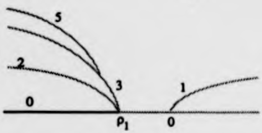


$\rho_1 > 0$

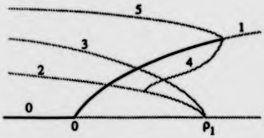


H : (f)<0, (g)<0

$\rho_1 < 0$



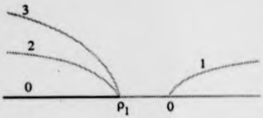
$\rho_1 > 0$



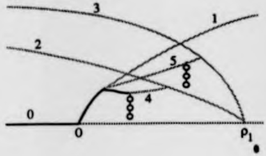
I: (b)<0, (c)<0, (d)<0, (e)<0

I^+ : (f)>0, (g)>0

$\rho_1 < 0$

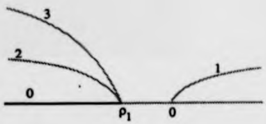


$\rho_1 > 0$

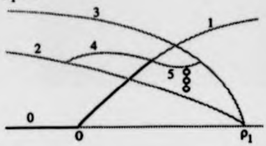


I^0 : (f)>0, (g)<0

$\rho_1 < 0$

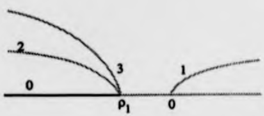


$\rho_1 > 0$

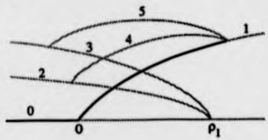


I^- : (f)<0, (g)<0

$\rho_1 < 0$



$\rho_1 > 0$



B.1 Breaking the S_2-symmetry

We break the S_2-symmetry of the $Z_2 \oplus D_4$-unfolding H by introducing the parameter $\tilde{\rho}_2$ as

$$\begin{aligned}
G_1(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1, \tilde{\rho}_2) &= [\epsilon_1 x^2 + \tilde{n}_1(y_1^2 + y_2^2) + \epsilon_2\lambda]x \\
G_2(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1, \tilde{\rho}_2) &= [\tilde{n}_2 x^2 + \tilde{n}_3(y_1^2 + y_2^2) + \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda + \tilde{\rho}_1 + \tilde{\rho}_2]y_1 \\
G_3(x, y_1, y_2, \lambda, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1, \tilde{\rho}_2) &= [\tilde{n}_2 x^2 + \tilde{n}_3(y_1^2 + y_2^2) - \epsilon_3(y_2^2 - y_1^2) + \epsilon_4\lambda + \tilde{\rho}_1 - \tilde{\rho}_2]y_2,
\end{aligned}$$

where $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{\rho}_1, \tilde{\rho}_2)$ varies on a neighbourhood of $(n_1, n_2, n_3, 0, 0)$. Now the branching equations and eigenvalues are

Label	Branching equations	Signs of eigenvalues
(1)	$\lambda = -\epsilon_1\epsilon_2 x^2$	ϵ_1 $n_2 x^2 + \epsilon_4\lambda + \tilde{\rho}_1 + \tilde{\rho}_2$ $n_2 x^2 + \epsilon_4\lambda + \tilde{\rho}_1 - \tilde{\rho}_2$
(2.1)	$\lambda = -\epsilon_4(\tilde{\rho}_1 + \tilde{\rho}_2) - \epsilon_4(n_3 - \epsilon_3)y_1^2$	$n_1 y_1^2 + \epsilon_2\lambda$ $n_3 - \epsilon_3$ $(n_3 + \epsilon_3)y_1^2 + \epsilon_4\lambda + \tilde{\rho}_1 - \tilde{\rho}_2$
(2.2)	$\lambda = -\epsilon_4(\tilde{\rho}_1 - \tilde{\rho}_2) - \epsilon_4(n_3 - \epsilon_3)y_2^2$	$n_1 y_2^2 + \epsilon_2\lambda$ $(n_3 + \epsilon_3)y_2^2 + \epsilon_4\lambda + \tilde{\rho}_1 + \tilde{\rho}_2$ $n_3 - \epsilon_3$
(4.1)	$\lambda = -\epsilon_1\epsilon_2 x^2 - \epsilon_2 n_1 y_1^2$ $= -\epsilon_4(\tilde{\rho}_1 + \tilde{\rho}_2) - \epsilon_4 n_2 x^2 - \epsilon_4(n_3 - \epsilon_3)y_1^2$	$\mathrm{tr} = \epsilon_1 x^2 + (n_3 - \epsilon_3)y_1^2$ $\det = [\epsilon_1(n_3 - \epsilon_3) - n_1 n_2]x^2 y_1^2$ $n_2 x^2 + (n_3 + \epsilon_3)y_1^2 + \epsilon_4\lambda + \tilde{\rho}_1 - \tilde{\rho}_2$
(4.2)	$\lambda = -\epsilon_1\epsilon_2 x^2 - \epsilon_2 n_1 y_2^2$ $= -\epsilon_4(\tilde{\rho}_1 - \tilde{\rho}_2) - \epsilon_4 n_2 x^2 - \epsilon_4(n_3 - \epsilon_3)y_2^2$	$\mathrm{tr} = \epsilon_1 x^2 + (n_3 - \epsilon_3)y_2^2$ $n_2 x^2 + (n_3 + \epsilon_3)y_2^2 + \epsilon_4\lambda + \tilde{\rho}_1 + \tilde{\rho}_2$ $\det = [\epsilon_1(n_3 - \epsilon_3) - n_1 n_2]x^2 y_2^2$
(6)	$\lambda = -\epsilon_4(\tilde{\rho}_1 + \tilde{\rho}_2) - \epsilon_4(n_3 - \epsilon_3)y_1^2 - \epsilon_4(n_3 + \epsilon_3)y_2^2$ $= -\epsilon_4(\tilde{\rho}_1 - \tilde{\rho}_2) - \epsilon_4(n_3 + \epsilon_3)y_1^2 - \epsilon_4(n_3 - \epsilon_3)y_2^2$	$n_1(y_1^2 + y_2^2) + \epsilon_2\lambda$ $\mathrm{tr} = n_3 - \epsilon_3$ $\det = -n_3\epsilon_3$
(7)	$\lambda = -\epsilon_1\epsilon_2 x^2 - n_1\epsilon_2(y_1^2 + y_2^2)$ $= -\epsilon_4(\tilde{\rho}_1 + \tilde{\rho}_2) - n_2\epsilon_4 x^2 - \epsilon_4(n_3 - \epsilon_3)y_1^2$ $-\epsilon_4(n_3 + \epsilon_3)y_2^2$ $= -\epsilon_4(\tilde{\rho}_1 - \tilde{\rho}_2) - n_2\epsilon_4 x^2 - \epsilon_4(n_3 + \epsilon_3)y_1^2$ $-\epsilon_4(n_3 - \epsilon_3)y_2^2$	

We keep the assumptions

$$\epsilon_2 = \epsilon_4 = -1$$

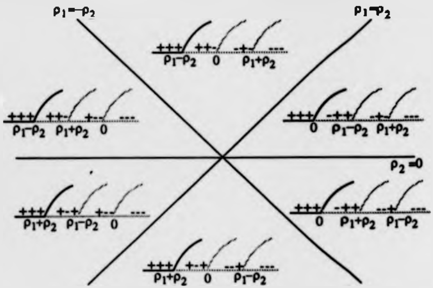
and

$$\epsilon_1 = \epsilon_3 = 1.$$

The values of λ where primary bifurcations occur are as in the table below. Again, given the assumptions on the ϵ_j, branch (1) is always supercritical and the criticality of branches (2.1) and (2.2) depends on the sign of expression (c).

From	To	Critical λ	Supercritical iff
(0)	(1)	$\lambda_{01} = 0$	
(0)	(2.1)	$\lambda_{02.1} = \rho_1 + \rho_2$	$(c) > 0$
(0)	(2.2)	$\lambda_{02.2} = \rho_1 - \rho_2$	$(c) > 0$

If (c) $>$ 0, all primary bifurcations are supercritical and the primary branches depend on ρ_1, ρ_2 as follows:



If $(c) < 0$, the bifurcations at $\rho_1 \pm \rho_2$ became subcritical. Secondary bifurcations are at the following values of λ if the associated conditions are satisfied.

From	To	Critical λ	Bifurcation iff Supercritical iff
(1)	(4.1)	$\lambda_{14.1} = \lambda_{01} + \frac{1}{(a)}(\lambda_{02.1} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{02.1} - \lambda_{01}) = \mathrm{sgn}(a)$ $\mathrm{sgn}(\lambda_{02.1} - \lambda_{01}) = \mathrm{sgn}(g)$
(1)	(4.2)	$\lambda_{14.2} = \lambda_{01} + \frac{1}{(a)}(\lambda_{02.2} - \lambda_{01})$	$\mathrm{sgn}(\lambda_{02.2} - \lambda_{01}) = \mathrm{sgn}(a)$ $\mathrm{sgn}(\lambda_{02.2} - \lambda_{01}) = \mathrm{sgn}(g)$
(2.1)	(4.1)	$\lambda_{24.1} = \lambda_{02.1} + \frac{(c)}{(e)}(\lambda_{01} - \lambda_{02.1})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{02.1}) = \mathrm{sgn}(e)$ $\mathrm{sgn}(\lambda_{01} - \lambda_{02.1}) = \mathrm{sgn}(g)$
(2.1)	(6)	$\lambda_{26.1} = \lambda_{02.1} - \frac{(c)}{2}(\lambda_{02.2} - \lambda_{02.1})$	$\lambda_{02.2} - \lambda_{02.1} < 0$ $(b) > 0$
(2.2)	(4.2)	$\lambda_{24.2} = \lambda_{02.2} + \frac{(c)}{(e)}(\lambda_{01} - \lambda_{02.2})$	$\mathrm{sgn}(\lambda_{01} - \lambda_{02.2}) = \mathrm{sgn}(e)$ $\mathrm{sgn}(\lambda_{01} - \lambda_{02.2}) = \mathrm{sgn}(g)$
(2.2)	(6)	$\lambda_{26.2} = \lambda_{02.2} - \frac{(c)}{2}(\lambda_{02.1} - \lambda_{02.2})$	$\lambda_{02.1} - \lambda_{02.2} < 0$ $(b) > 0$

Tertiary bifurcations occur at the following values of λ if the associated conditions are satisfied.

From	To	Critical λ	Bifurcation iff
(4.1)	(7)	$\lambda_{47.1} = \lambda_{14.1} - \frac{(g)}{2}(\lambda_{14.2} - \lambda_{14.1})$ $= \lambda_{24.1} + \frac{(e)}{(a)(c)}(\lambda_{26.1} - \lambda_{24.1})$	$\mathrm{sgn}(\lambda_{02.1} - \lambda_{01}) = \mathrm{sgn}(a)$ or $-\mathrm{sgn}(e)$ and $\mathrm{sgn}(\lambda_{02.1} - \lambda_{01}) = \mathrm{sgn}(a) \Rightarrow$ $\Rightarrow \mathrm{sgn}(\lambda_{14.2} - \lambda_{14.1}) = -\mathrm{sgn}(a)$ and $\mathrm{sgn}(\lambda_{01} - \lambda_{02.1}) = \mathrm{sgn}(e) \Rightarrow$ $\Rightarrow \mathrm{sgn}(\lambda_{26.1} - \lambda_{24.1}) = \mathrm{sgn}(a)\mathrm{sgn}(c)\mathrm{sgn}(e)$
(4.2)	(7)	$\lambda_{47.2} = \lambda_{14.2} - \frac{(g)}{2}(\lambda_{14.1} - \lambda_{14.2})$ $= \lambda_{24.2} + \frac{(e)}{(a)(c)}(\lambda_{26.2} - \lambda_{24.2})$	$\mathrm{sgn}(\lambda_{02.2} - \lambda_{01}) = \mathrm{sgn}(a)$ or $-\mathrm{sgn}(e)$ and $\mathrm{sgn}(\lambda_{02.2} - \lambda_{01}) = \mathrm{sgn}(a) \Rightarrow$ $\Rightarrow \mathrm{sgn}(\lambda_{14.1} - \lambda_{14.2}) = -\mathrm{sgn}(a)$ and $\mathrm{sgn}(\lambda_{01} - \lambda_{02.2}) = \mathrm{sgn}(e) \Rightarrow$ $\Rightarrow \mathrm{sgn}(\lambda_{26.2} - \lambda_{24.2}) = \mathrm{sgn}(a)\mathrm{sgn}(c)\mathrm{sgn}(e)$
(6)	(7)	$\lambda_{67} = \lambda_{26.1} + \frac{(b)(g)}{(d)(c)}(\lambda_{24.1} - \lambda_{26.1})$ $= \lambda_{26.2} + \frac{(b)(g)}{(d)(c)}(\lambda_{24.2} - \lambda_{26.2})$	$\lambda_{02.2} - \lambda_{02.1} < 0 \Rightarrow \mathrm{sgn}(\lambda_{24.1} - \lambda_{26.1}) =$ $= \mathrm{sgn}(b)\mathrm{sgn}(c)\mathrm{sgn}(d)\mathrm{sgn}(g)$ $\lambda_{02.1} - \lambda_{02.2} < 0 \Rightarrow \mathrm{sgn}(\lambda_{24.2} - \lambda_{26.2}) =$ $= \mathrm{sgn}(b)\mathrm{sgn}(c)\mathrm{sgn}(d)\mathrm{sgn}(g)$

Recall that the bifurcation diagrams for $\rho_2 = 0$ and $(a) > 0$ are shown before. The picture can be completed for $\rho_2 \neq 0$ directly from the tables above. This is not done here. We end this appendix by noting that it is enough to draw the bifurcation diagrams for either $\rho_2 > 0$ or $\rho_2 < 0$ since these two cases are S_2-conjugate.

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